LINEAR STATISTICAL INFERENCE BASED ON L-ESTIMATORS

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1. Introduction

Consider the linear model

$$(1.1) \qquad \underset{\sim}{Y} = \underset{\sim}{X}\underset{\sim}{\beta} + \underset{\sim}{E}$$

where $\underset{\sim}{Y} = (Y_1,\ldots,Y_n)'$ is a vector of independent observations, $\underset{\sim}{X}$ is a known $(n \times p)$-matrix, $\underset{\sim}{\beta} = (\beta_1,\ldots,\beta_p)'$ is an unknown parameter and $\underset{\sim}{E} = (E_1,\ldots,E_n)'$ is a vector of independent errors identically distributed according to a distribution function (d.f.) F, which is either unknown or only partially known. We want to find an estimator for $\underset{\sim}{\beta}$ which has high efficiency under normal d.f. F but which is able to endure mild perturbations from normality. The latter requirement is not satisfied by the classical least-squares estimator (LSE).

In the location submodel ($p=1$, $x_{11}=1$, $x_{ij}=0$; $i=1,\ldots,n$, $j=2,\ldots,n$), three broad classes of robust estimators, less sensitive to deviations in the distribution shape, namely M- , R- and L-estimators, took the most interest in the statistical practice. From these three classes, M- and R-estimators extend in a straightforward way to the linear model. Their detailed study may be found, e.g., in Huber(1981).

L-estimators have a simple structure in the location model, being just linear combinations of order statistics, yet they have good efficiency and robustness properties. Well-known is the sample median $\tilde{Y}_n$, the α-trimmed mean,

$$(1.2) \qquad L_n(\alpha) = \frac{1}{n-2\,[n\alpha]} \sum_{i=[n\alpha]+1}^{n-[n\alpha]} Y_{n:i},$$

or the α-Winsorized mean,

$$(1.3) \qquad L_n^{*}(\alpha) = n^{-1}\left\{ [n\alpha]\, Y_{n:[n\alpha]} + \sum_{i=[n\alpha]+1}^{n-[n\alpha]} Y_{n:i} + [n\alpha]\, Y_{n:n-[n\alpha]+1} \right\}$$

where $0 < \alpha < 1/2$, $[x]$ is the largest integer k satisfying $k \le x$ and $Y_{n:1} \le \ldots \le Y_{n:n}$ are the order statistics corresponding to $Y_1,\ldots,Y_n$. An extension of L-estimators to the linear model is not as straightforward as in the case of other estimators. One possible extension was suggested by Bickel (1973); but Bickel's estimators are complicated and are not invariant to the reparametrization of design.

Koenker and Bassett (1978) introduced the concept of regression quantile as an extension of the sample quantile to the linear model. This concept seems to provide the right basis for the inference in linear models based on the ordered residuals. While the earlier attempts ordered the residuals from some preliminary estimate, the present procedures order the residuals from the regression quantiles and hence are not effected by the choice of initial estimator.

The present paper provides a review of some estimators and tests based on regression quantiles. The area is under development and thus the review is not exhaustive; but even the partial results indicate that the procedures are not only theoretically interesting but that they are also easily applicable to the practical problems.

2. Regression quantiles and trimmed least-squares estimators

Let $Y_1,\ldots,Y_n$ be independent observations, Y_i distributed according to the d.f. $F(y - \sum_{j=1}^{p} x_{ij}\beta_j)$, $i=1,\ldots,n$, where F is an absolutely continuous d.f. and $\underset{\sim}{X} = (x_{ij})_{i=1,\ldots,n}^{j=1,\ldots,p}$ is a known design matrix. Fix $\alpha \in (0,1)$ and consider the function

(2.1)
$$\rho_\alpha(x) = \begin{cases} \alpha x & \text{if } x \geq 0 \\ (\alpha-1)x & \text{if } x < 0. \end{cases}$$

Denote $\hat{\underset{\sim}{\beta}}(\alpha) = (\hat{\beta}_1(\alpha),\ldots,\hat{\beta}_p(\alpha))'$ the solution of the minimization problem

(2.2)
$$\sum_{i=1}^{n} \rho_\alpha(Y_i - \sum_{j=1}^{p} x_{ij} t_j) : = \min$$

with respect to $\underset{\sim}{t} \in R^p$. Notice that, in the location submodel, solution of (2.2) leads to the sample α-quantile. Koenker and Bassett (1978) called $\hat{\underset{\sim}{\beta}}(\alpha)$ the regression α-quantile.

If $\underset{\sim}{X}$ contains an intercept, i.e., $x_{i1}=1$, $i=1,\ldots,n$, if

(2.3)
$$\frac{1}{n} \underset{\sim}{X}'\underset{\sim}{X} \rightarrow \underset{\sim}{Q} \quad (\text{positive definite } pxp\text{ -matrix})$$

as $n \to \infty$, and if F has positive and finite derivative in a neighborhood of $F^{-1}(\alpha)$, then (Koenker and Bassett (1978))

(2.4)
$$\mathcal{L}\{n^{1/2}(\hat{\underset{\sim}{\beta}}(\alpha) - \underset{\sim}{\beta}(\alpha))\} \rightarrow N_p(\underset{\sim}{0}, [\alpha(1-\alpha)/f^2(F^{-1}(\alpha))]\underset{\sim}{Q}^{-1})$$

as $n \to \infty$, where

(2.5)
$$\underset{\sim}{\beta}(\alpha) = (\beta_1 + F^{-1}(\alpha), \beta_2,\ldots,\beta_p)'.$$

The simultaneous asymptotic distribution of several regression quantiles is analogous to that of several sample quantiles : if $\underset{\sim}{\hat\xi} = (\hat{\underset{\sim}{\beta}}(\alpha_1),\ldots,\hat{\underset{\sim}{\beta}}(\alpha_m))'$ and $\underset{\sim}{\xi} = (\underset{\sim}{\beta}(\alpha_1),\ldots,\underset{\sim}{\beta}(\alpha_m))'$ then the asymptotic distribution of $n^{1/2}(\underset{\sim}{\hat\xi} - \underset{\sim}{\xi})$ is mp-dimensional normal $N_{mp}(\underset{\sim}{0}, \underset{\sim}{\Omega} \times \underset{\sim}{Q}^{-1})$ where $\underset{\sim}{\Omega} = (\omega_{ij})_{i,j=1}^{m}$ with

(2.6)
$$\omega_{ij} = \alpha_i(1-\alpha_j)/f(F^{-1}(\alpha_i))f(F^{-1}(\alpha_j)), \quad i,j=1,\ldots,m,$$

provided F has positive bounded derivative in a neighborhood of $F^{-1}(\alpha_1),\ldots,$
$\ldots,F^{-1}(\alpha_m)$.

If we put $\alpha = 1/2$, we get an l_1-estimator, which is an extension of the sample median to the regression model.

Koenker and Bassett (1978) also suggested the trimmed_least-squares_estimator (trimmed LSE) in the following way : fix α_1, α_2, $0 < \alpha_1 < \alpha_2 < 1$, and calculate the regression quantiles $\hat{\underset{\sim}{\beta}}(\alpha_1)$ and $\hat{\underset{\sim}{\beta}}(\alpha_2)$. The solution of (2.2) needs not to be uniquely determined but we could always determine a rule which selects one of the possible solutions (the choice of a single solution does not effect the asymptotic results). Let $\underset{\sim}{A}$ be the diagonal nxn matrix with the diagonal

(2.7)
$$a_{ii} = a_i = \begin{cases} 0 & \text{if } Y_i \leq \sum_{j=1}^{p} x_{ij}\hat{\beta}_j(\alpha_1) \text{ or } Y_i \geq \sum_{j=1}^{p} x_{ij}\hat{\beta}_j(\alpha_2) \\ 1 & \text{otherwise}, \quad i=1,\ldots,n. \end{cases}$$

Trimm-off all Y_i with $a_i=0$, $i=1,\ldots,n$ and calculate the ordinary LSE from the remaining observations. The resulting estimator $\underset{\sim}{L}_n(\alpha_1,\alpha_2)$, which could be also expressed as

(2.8)
$$\underset{\sim}{L}_n(\alpha_1, \alpha_2) = (\underset{\sim}{X}'\underset{\sim}{A}\,\underset{\sim}{X})^- (\underset{\sim}{X}'\underset{\sim}{A}\,\underset{\sim}{Y})$$

$[\underset{\sim}{C}^-$ denotes the generalized inverse of $\underset{\sim}{C}]$ is called the trimmed least-squares estimator of β .

This estimator was later studied in details by Ruppert and Caroll (1980) who confirmed the asymptotic distribution conjectured by Koenker and Bassett (1978). Let us formulate their result as a theorem.

THEOREM 1. (Ruppert and Caroll (1980)). Let $Y_1,\ldots,Y_n$ be independent random variables , Y_1 distributed according to the d.f. $F(y - \sum_{j=1}^{p} x_{ij} \beta_j)$; let F and $\underset{\sim}{X} = (x_{ij}) \begin{smallmatrix} j=1,\ldots,p \\ i=1,\ldots,n \end{smallmatrix}$ satisfy the following conditions :

1° F has a continuous density f such that $f(y) > 0$ on the support of F.

2° $x_{i1} = c$, $i=1,..,n$ and $\sum_{i=1}^{n} x_{ij} = 0$, $j=2,..,p$.

3° $\lim_{n\to\infty} \max_{1\le i \le n} \left\{ n^{-1/2} |x_{ij}| \right\} = 0$, $j=1,\ldots,p$.

4° There exists a positive definite $\underset{\sim}{Q}$ such that $\lim_{n\to\infty} n^{-1} \underset{\sim}{X}'\underset{\sim}{X} = \underset{\sim}{Q}$.

Then

(2.9)
$$\underset{\sim}{L}_n(\alpha_1, \alpha_2) - \underset{\sim}{\beta} = n^{-1}(\alpha_2 - \alpha_1)^{-1} \underset{\sim}{Q}^{-1} \sum_{i=1}^{n} \underset{\sim}{x}_i \psi(E_i) + o_p(n^{-1/2}),$$

as $n \to \infty$, where $\underset{\sim}{x}_i = (x_{i1},\ldots,x_{ip})'$, and

(2.10)
$$\psi(x) = \begin{cases} F^{-1}(\alpha_1) & \text{if} \quad x < F^{-1}(\alpha_1) \\ x & \text{if} \quad F^{-1}(\alpha_1) \le x \le F^{-1}(\alpha_2) \\ F^{-1}(\alpha_2) & \text{if} \quad F^{-1}(\alpha_2) < x. \end{cases}$$

Consequently,

(2.11) $\mathcal{L}\left\{ n^{1/2}(\underset{\sim}{L}_n(\alpha_1, \alpha_2) - \underset{\sim}{\beta}) \right\} \to N_p(\underset{\sim}{0}, \sigma^2(\alpha_1, \alpha_2, F) \underset{\sim}{Q}^{-1})$

where

(2.12) $\sigma^2(\alpha_1, \alpha_2, F) = (\alpha_2 - \alpha_1)^{-2} \Big\{ \int_{\alpha_1}^{\alpha_2} (F^{-1}(u) - \delta)^2 du + \alpha_1 (F^{-1}(\alpha_1) - \delta)^2 + (1 - \alpha_2)(F^{-1}(\alpha_2) - \delta)^2$
$- \left[\alpha_1 (F^{-1}(\alpha_1) - \delta) + (1 - \alpha_2)(F^{-1}(\alpha_2) - \delta) \right]^2 \Big\}$

and

(2.13) $\delta = \delta(\alpha_1, \alpha_2, F) = (\alpha_2 - \alpha_1)^{-1} \int_{\alpha_1}^{\alpha_2} F^{-1}(u) du.$

Notice that $\sigma^2(\alpha_1, \alpha_2, F)$ is the asymptotic variance of the trimmed mean in the location case. If, besides the assumptions of Theorem 1, we assume that F is symmetric and put $\alpha_1 = 1 - \alpha_2$ we see that $\delta = 0$ and $\sigma^2(\alpha, F)$ takes on a simpler form. Combining the representation (2.9) with Theorem 4.1 of Jurečková (1977), we get

(2.14) $n^{1/2} \left\| \underset{\sim}{L}_n(\alpha) - \underset{\sim}{M}_n \right\| \xrightarrow{P} 0,$ as $n \to \infty$

where $\underset{\sim}{M}_n$ is the M-estimator defined as a solution of the system of equations

(2.15) $\sum_{i=1}^{n} x_{ik} \psi(Y_i - \sum_{j=1}^{p} x_{ij} t_j) = 0$, $k=1,\ldots,p$

with ψ of (2.10). The M- estimator generated by the function

$$(2.16) \qquad \psi_k(x) = \begin{cases} x & \text{if } |x| \leq k \\ k.\text{sign } x & \text{if } |x| > k \end{cases}$$

with $k > 0$ is known as Huber's estimator. Hence, we may conclude that the class of trimmed LSE's is asymptotically equivalent to the class of Huber's estimators.

Comparing with $\underset{\sim}{M}_n$, the trimmed LSE is scale-equivariant, i.e., $\underset{\sim}{L}_n(c\, Y_1,...,\, ..., c\, Y_n) = c\, \underset{\sim}{L}_n(Y_1,...,Y_n)$, $c > 0$. This is not the case of Huber's M-estimator $\underset{\sim}{M}_n$: to make it scale equivariant, we should either supplement $\underset{\sim}{M}_n$ by an estimator of scale on we could alternatively apply the following adaptive procedure proposed by Jurečková and Sen (1984). Fix $\alpha \in (0, 1/2)$, calculate the regression quantiles $\underset{\sim}{\hat{\beta}}(\alpha)$ and $\underset{\sim}{\hat{\beta}}(1-\alpha)$ and put

$$(2.17) \qquad k_n(\alpha) = \tfrac{1}{2}(\hat{\beta}_1(1-\alpha) - \hat{\beta}_1(\alpha)).$$

Let $\underset{\sim}{M}_n^*$ be an estimator defined as a solution of (2.15) with $\psi = \psi_{k_n(\alpha)}$. Then $\underset{\sim}{M}_n^*$ is scale-equivariant and, under the conditions of Theorem 1 and for symmetric F, the asymptotic distribution of $\underset{\sim}{M}_n^*$ concides with that of (2.11)-(2.13) $\big[$with $\alpha_1 = \alpha$, $\alpha_2 = 1 - \alpha$, $\delta = 0 \big]$. Moreover, if we put

$$(2.18) \qquad \alpha = (1-\varepsilon)(1-\Phi(k)) + \tfrac{\varepsilon}{2}$$

with k satisfying

$$(2.19) \qquad 2(\varphi(k)/k) - 2\Phi(-k) = \varepsilon/(1-\varepsilon)$$

$\big[\Phi$ is the d.f. of $N(0,1)$ and $\varphi(x) = \frac{d\Phi}{dx}\big]$ then $\underset{\sim}{M}_n^*$ is the minimax estimator minimizing the maximum asymptotic variance over the family of ε-contaminated normal distributions,

$$(2.20) \qquad \mathscr{F} = \big\{ F = (1-\varepsilon)\Phi + \varepsilon H, \quad H \in \mathscr{H} \big\}$$

where $\mathscr{H}$ is the family of symmetric d.f.'s (cf. Huber (1964, 1981).

Ruppert and Caroll (1980) also derived the Bahadur representation of regression quantile $\underset{\sim}{\hat{\beta}}(\alpha)$: under the assumptions of Theorem 1,

$$(2.21) \quad \underset{\sim}{\hat{\beta}}(\alpha) - \underset{\sim}{\beta}(\alpha) = n^{-1}\big[f(F^{-1}(\alpha))\big]^{-1} \underset{\sim}{Q}^{-1} \sum_{i=1}^{n} \underset{\sim}{x}_i\, \psi_\alpha(E_i - F^{-1}(\alpha)) + o_p(n^{-1/2})$$

where

$$(2.22) \qquad \psi_\alpha(x) = \alpha - I[x < 0], \qquad x \in R^1.$$

Let us now consider a possible definition and properties of the trimmed LSE in a more general situation when some of the conditions of Theorem 1 are violated:

(i) If the design matrix $\underset{\sim}{X}$ does not satisfy 2° and hence there is no intercept, we suggest to extend $\underset{\sim}{X}$ by a column of units to

$$(2.23) \qquad \underset{\sim}{X}^* = (\underset{\sim}{1} \vdots \underset{\sim}{X})$$

where $\underset{\sim}{1} = (1,...,1)'$ is an $n \times 1$ vector; then we shall consider the extended model

$$(2.24) \qquad \underset{\sim}{Y} = \underset{\sim}{X}^* \underset{\sim}{\beta}^* + \underset{\sim}{E}$$

with $\underset{\sim}{\beta}^* = (\beta_0, \beta_1,..., \beta_p)'$; $\beta_0(=0)$ is an artificial intercept. We calculate the trimmed LSE $\underset{\sim}{L}_n(\alpha_1, \alpha_2) = (L_0^*, L_1^*,...,L_p^*)'$ for the extended model in the

Koenker-Bassett manner and define $L_n(\alpha_1, \alpha_2) = (L_1^*, \ldots, L_p^*)'$ as the trimmed LSE for the original model. Then

(2.25) $\quad \mathcal{L}\{n^{1/2}(L_n(\alpha_1, \alpha_2) - \beta)\} \to N_p(0, \sigma^2(\alpha_1, \alpha_2, F) \bar{Q}^{-1})$,

as $n \to \infty$, where $\sigma^2(\alpha_1, \alpha_2, F)$ concides with (2.12) and $\bar{Q} = (q_{jk})_{j,k=1}^p$ with

(2.26) $\quad \bar{q}_{jk} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k), \quad \bar{x}_j = \sum_{i=1}^n x_{ij}, \quad j=1,\ldots,p.$

The asymptotic distribution (2.25) and the representation of the type (2.21) in a more general setup and with the remaining term of order $O_p(n^{-3/4})$ are proved in Jurečková (1984).

(ii) As a special case, consider the situation that X satisfies

(2.27) $\quad \sum_{i=1}^n x_{ij} = 0, \quad j=1,\ldots,p.$

Then, if we defined $\hat{\beta}(\alpha)$ as a solution of (2.2) we should come to a surprising solution that the regression quantiles corresponding to various α are asymptotically undistinguishable from each other. More precisely, under 1^0, 3^0 and 4^0 of Theorem 1,

(2.28) $\quad \hat{\beta}(\alpha) - \beta = n^{-1}(f(0))^{-1} Q^{-1} \sum_{i=1}^n x_i \varphi_{\alpha_0}(E_i) + o_p(n^{-1/2})$

holds for $0 < \alpha < 1$, as $n \to \infty$, with $\alpha_0 = F(0)$ and φ_α of (2.22). This proposition is proved in Jurečková (1984), again strengthened to $O_p(n^{-3/4})$. We get as a consequence that the trimmed LSE is not well-defined in such situation. In such and similar cases, we recommend to use the approach described in (i).

(iii) Jurečková (1983a) introduced the Winsorized LSE, studied its asymptotic properties and found its M-estimator counterpart. For $\alpha \in (0, 1/2)$ the α-Winsorized LSE is defined as

(2.29) $\quad L_n^*(\alpha) = n^{-1}\left\{ [n\alpha] \hat{\beta}(\alpha) + (n-2\lceil n\alpha \rceil) L_n(\alpha) + [n\alpha] \hat{\beta}(1-\alpha) \right\}$

where $L_n(\alpha)$ is the α-trimmed LSE. The asymptotic properties of $L_n^*(\alpha)$ are analogous to those of the α-Winsorized mean in the location case.

An important aspect of every statistical procedure is that of its computation. A possible computation of regression quantiles and of the trimmed LSE was already considered by Koenker and Bassett. It turns out that the regression quantiles could be calculated by a modified linear programming algorithm and hence the calculation of the trimmed LSE (or that of the Winsorized LSE) merely requires the solution of a linear programming problem in addition to the usual least squares computation. Antoch (1984) elaborated a series of computer programs convenient for a direct practical use. An extensive Monte Carlo study by Antoch, Collomb and Hassani (1984) illustrates that the L-estimators behave well for non-normal distributions and reasonably well for normal distribution, comparing with the LSE.

3. Tests of linear and other hypotheses

The trimmed LSE can be used for testing the linear hypothesis in an analogous way as the classical tests use the ordinary LSE. The pertaining critical regions are based on the asymptotic distributions of the test statistics.

Ruppert and Caroll (1980) constructed a consistent estimator of $\sigma^2(\alpha_1, \alpha_2, F)$ of (2.12) of the form

(3.1) $\quad S_n^2 = (\alpha_2 - \alpha_1)^{-1} \Big\{ (n-p)^{-1} z_n^2 + \alpha_1 (\hat{\beta}_1(\alpha_1) - L_1)^2$

$\qquad + (1-\alpha_2)(\hat{\beta}_1(\alpha_2) - L_1)^2 - [\alpha_1(\hat{\beta}_1(\alpha_1) - L_1) + (1-\alpha_2)(\hat{\beta}_1(\alpha_2) - L_1)]^2 \Big\}$

where

(3.2) $\qquad z_n^2 = \underset{\sim}{Y}' \underset{\sim}{A} \big[\underset{\sim}{I}_p - \underset{\sim}{X}(\underset{\sim}{X}' \underset{\sim}{A} \underset{\sim}{X})^- \underset{\sim}{X} \big] \underset{\sim}{A} \underset{\sim}{Y} ;$

$\hat{\beta}_1(\alpha_1)$ and L_1 is the first component of $\hat{\underset{\sim}{\beta}}(\alpha_i)$, $i=1,2$ and of $\underset{\sim}{L}_n(\alpha_1, \alpha_2)$ respectively. The convergence

(3.3) $\qquad S_n^2 \xrightarrow{p} \sigma^2(\alpha_1, \alpha_2, F)$, as $\quad n \to \infty$

was proved by Ruppert and Caroll (1980) under the conditions of Theorem 1 and by Jurečková (1984) under more general conditions including the polynomial regression.

To illustrate some tests based on the trimmed LSE, it is convenient to rewrite the linear model (1.1) in the form

(3.4) $\qquad \underset{\sim}{Y} = \beta_1 \cdot \underset{\sim}{1} + \overline{\underset{\sim}{X}} \overline{\underset{\sim}{\beta}} + \underset{\sim}{E}$

where $\underset{\sim}{1} = (1,\dots,1)'$ is $(n \times 1)$ vector of units, $\overline{\underset{\sim}{X}} = (x_{ij})_{i=1,\dots,n}^{j=2,\dots,p}$ and $\overline{\underset{\sim}{\beta}} = (\beta_2,\dots,\beta_p)'$. Consider the hypothesis

$\qquad H_o : \beta_j = 0$. $\qquad j = p-m+1,\dots,p \qquad (1 \leq m \leq p-1)$.

Let $\underset{\sim}{L} = \underset{\sim}{L}_n(\alpha_1, \alpha_2)$ denote the trimmed LSE of $\underset{\sim}{\beta} = (\beta_1,\dots,\beta_p)'$ and let $\overline{\underset{\sim}{L}} = \overline{\underset{\sim}{L}}_n(\alpha_1, \alpha_2)$ denote the same calculated under H_o. They we may test H_o with the aid of statistic

(3.5) $\qquad T_n = (\underset{\sim}{L} - \overline{\underset{\sim}{L}})' \underset{\sim}{X}' \underset{\sim}{X} (\underset{\sim}{L} - \overline{\underset{\sim}{L}}) / S_n^2.$

Under the assumptions 1° and 4° of Theorem 1, provided

(3.6) $\qquad \underset{1 \leq i \leq n}{\max} |x_{ij}| = O(n^{1/4})$, as $\quad n \to \infty$, for $j=2,\dots,p$

the test with the critical region

(3.7) $\qquad T_n \geq \chi_m^2(\gamma)$, $\qquad 0 < \gamma < 1$

has an asymptotic size γ, as $n \to \infty$ $[\chi_m^2(\gamma)$ is $(1-\gamma)$-quantile of χ^2 distribution with m degrees of freedom]; moreover, its Pitman efficiency with respect to the classical F-test coincides with the ARE of the trimmed LSE to the classical LSE. This is proved in Jurečková (1983b) for the case of polynomial regression but the proofs apply to the present setup as well.

Ruppert and Caroll (1980) mention the possible tests of $H_o : \underset{\sim}{K} \overline{\underset{\sim}{\beta}} = \underset{\sim}{c}$ with $\underset{\sim}{K}$ being an $1 \times (p-1)$ matrix of the rank 1, under the conditions 1°-4°. The latter conditions do not cover the polynomial regression but hopefully the tests can be extended to more general cases with the aid of metodology developed in Jurečková (1983b, 1984).

Parallel tests based on l_1-estimator were developed by Koenker and Bassett (1982a); the tests represent analogous of classical Wald's, likelihood ratio and Lagrange multiplier test, respectively. In another paper, Koenker and Bassett(1982b) propose a test of heteroscedasticity based on regression quantiles, more precisely, a test of hypothesis $H_1 : \underset{\sim}{\gamma} = \underset{\sim}{0}$ in the model

(3.8) $\qquad Y_i = \sum_{j=1}^{p} x_{ij} \beta_j + \sigma_i E_i,$ $\qquad i=1,\dots,n$

with

$$(3.9) \qquad \delta_i = 1 + h \left(\sum_{k=1}^{r} z_{ik} \gamma_k \right), \qquad i=1,\dots,n$$

where x_{ij} and z_{ik} $(i=1,\dots,n;\ j=1,\dots,p;\ k=1,\dots,r)$ are known constants, $\beta = (\beta_1,\dots,\beta_p)'$ and $\gamma = (\gamma_1,\dots,\gamma_r)'$ unknown parameters and h is a known, smooth positive function. The power of the test based on regression quantiles is less sensitive to the kurtosis of E_i (and thus to possibly long-tailed distributions) than the tests earlier proposed for normally distributed E_i.

References

Antoch,J. (1984). Collection of programs for robust regression analysis. Charles University Prague.

Antoch,J., Collomb,G. and Hassani, S.(1984). Robustness in parametric and nonparametric regression: An investigation by computer simulation. COMPSTAT

Bickel, P.J.(1973). On some analogues to linear combinations of order statistics in the linear model. Ann.Statist.1, 597-616.

Huber,P.J. (1964). Robust estimation of a location parameter. Ann.Math.Statist. 35, 73-101.

Huber,P.J. (1981). Robust Statistics. J. Wiley, New York.

Jurečková,J. (1977). Asymptotic relations of M-estimates and R-estimates in linear regression model. Ann.Statist. 5, 464-472.

Jurečková,J. (1983a). Winsorized least-squares estimator and its M-estimator counterpart. Contributions to Statistics: Essays in Honour of Norman L.Johnson (ed. P.K.Sen), pp.237-245. North-Holland, Amsterdam.

Jurečková,J. (1983b). Trimmed polynomial regression. Comment.Math.Univ.Carolinae, 24, 597-607.

Jurečková,J. (1983c). Robust estimators of location and regression parameters and their second order asymptotic relations. Trans. 9th Prague Conf.on Inform. Theory, Random Proc.and Statist.Desic. Functions, pp.19-32.Academia,Prague.

Jurečková,J.(1984). Regression quantiles and trimmed least-squares estimator under a general design. Kybernetika 20, No 5

Jurečková,J. and Sen,P.K.(1984). On adaptive scale-equivariant M-estimators in linear models. Statistics & Desicions 2

Koenker,R.(1982). Robust methods in econometrics. Economic Reviews 1, 213-255.

Koenker,R. and Bassett,G., Jr.(1978). Regression quantiles. Econometrica 46,33-50.

Koenker,R. and Bassett,G.,Jr.(1982a). Tests of linear hypotheses and l_1 estimation. Econometrica 50, 1577-1583.

Koenker,R. and Bassett,G., Jr.(1982b). Robust tests for heteroscedasticity based on regression quantiles. Econometrica 50, 43-62.

Ruppert, D. and Carroll, R.J.(1980). Trimmed least squares estimation in the linear model. Journ.Amer.Statist.Assoc., 75, 828-838.