Application of permutation principle in multiple structural change test

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Abstract

We deal with an F type test for detection of changes in multiple linear regresion models. Approximations to critical values are usually obtained via the limit distribution of the test statistic under the null hypothesis. Here we explore another possibility - a method based on the application of the permutation principle.

1. Model

For the j-th segment, $j = 1, \dots, m+1$

$$y_t = \boldsymbol{z}_t' \boldsymbol{\delta}_j + e_t \qquad t = t_{j-1}+1, \dots, t_j \qquad (1)$$

Convention $t_0 = 1$, $t_{m+1} = n$

- $t_1, t_2, \dots, t_m$ — change points, often unknown
- $y_1, \dots, y_n$ — observed dependent variables
- $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n$ — vectors of regressors
- $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_{m+1}$ — vectors of regression coefficients
- $e_1, \dots, e_n$ — errors

2. Assumptions

- The errors are independent and identically distributed with zero mean, nonzero variance σ^2 and finite moment $\mathsf{E}|e_t|^{2+\Delta}$ with some $\Delta > 0$.
- The errors e_t are independent of the regressors $\boldsymbol{z}_s$ for all t and s.
- The regressors $\boldsymbol{z}_t$ are nontrending, i.e. $(\boldsymbol{Z}_j' \boldsymbol{Z}_j)/(t_j - t_{j-1})$ converges in probability to some finite positive definite matrix $\boldsymbol{C}$ as $t_j - t_{j-1} \to \infty$, $j = 1, \dots, m+1$, where $\boldsymbol{Z}_j = (\boldsymbol{z}_{t_{j-1}+1}, \dots, \boldsymbol{z}_{t_j})'$.
- $t_j = [n\lambda_j]$, $j = 1, \dots, m$, $0 = \lambda_0 < \lambda_1 < \dots < \lambda_{m+1} = 1$.

3. Estimation

- Least squares principle
- The minimal length of a segment is $h \geq q$ (q regressors in each segment):

$$T_h = \{(t_1, \dots, t_m) : t_{j+1} - t_j \geq h, \forall j = 0, \dots, m\}$$

- Minimal sum of squared residuals (SSR) for a given partition $(t_1, \dots, t_m)$:

$$S_n(t_1, \dots, t_m) \equiv \sum_{j=1}^{m+1} \min_{\boldsymbol{\delta}_j} \sum_{t=t_{j-1}+1}^{t_j} (y_t - \boldsymbol{z}_t' \boldsymbol{\delta}_j)^2$$

- The change points are estimated as

$$(\hat{t}_1, \dots, \hat{t}_m) = \arg\min_{t_1, \dots, t_m \in T_h} S_n(t_1, \dots, t_m)$$

4. F type test

- Hypotheses
  - H_0: $m = 0$ (no change)
  - H_A: $m = k$ (k changes)
- Test statistic

$$\sup F_n^{\varepsilon}(k, q) \equiv \frac{SSR_0 - SSR_k}{kq\, \hat{\sigma}_k^2} \qquad (2)$$

$SSR_0 = \min_{\boldsymbol{\delta}} \sum_{t=1}^{n} (y_t - \boldsymbol{z}_t' \boldsymbol{\delta})^2$ — minimal SSR under H_0

$SSR_k = S_n(\hat{t}_1, \dots, \hat{t}_k)$ — minimal SSR under H_A

$\hat{\sigma}_k^2 = SSR_k / (n - (k+1)\, q)$ — consistent estimator of σ^2

- The limit distribution of (2) under H_0 (derived in [1]) depends on parameter $\varepsilon = h/n$, number of changes k under H_A, number of regressors q. As $\varepsilon \to 0$, the critical values of (2) diverge to infinity.

5. Permutation principle

- Under H_0 errors e_t are iid $\to$ $(e_1, \dots, e_n)$ have the same distribution as $(e_{R_1}, \dots, e_{R_n})$, where $\boldsymbol{R} = (R_1, \dots, R_n)$ is a random permutation of $(1, \dots, n)$.
- e_t unknown $\to$ replaced by their estimators under H_0 - residuals $\hat{e}_t$ from model (1) with $m = 0$.
- For each random permutation $\boldsymbol{R}_1, \dots, \boldsymbol{R}_N$, $N << n!$ and N is large enough, calculate permutational version of (2)

$$\sup F_n^{\varepsilon}(k, q; \boldsymbol{R}) = \frac{SSR_0(\boldsymbol{R}) - SSR_k(\boldsymbol{R})}{kq\, \hat{\sigma}_k^2(\boldsymbol{R})} \qquad (3)$$

where observations y_t are replaced by permutated residuals $\hat{e}_{R_t}$.

- Calculate the empirical distribution of (3) and the corresponding empirical quantiles.
- The conditional limit distribution of (3), given $y_1, \dots, y_n$ (the data may follow H_0 or the alternatives) coincides with the limit distribution of (2) under H_0 (the proof is sketched in [4] assuming known change points under H_A).
- Therefore the calculated empirical quantiles serve as the approximations to the critical values corresponding to the test (2).
- Some simulation results in Table 1
- Example of simulated data in Figure 1

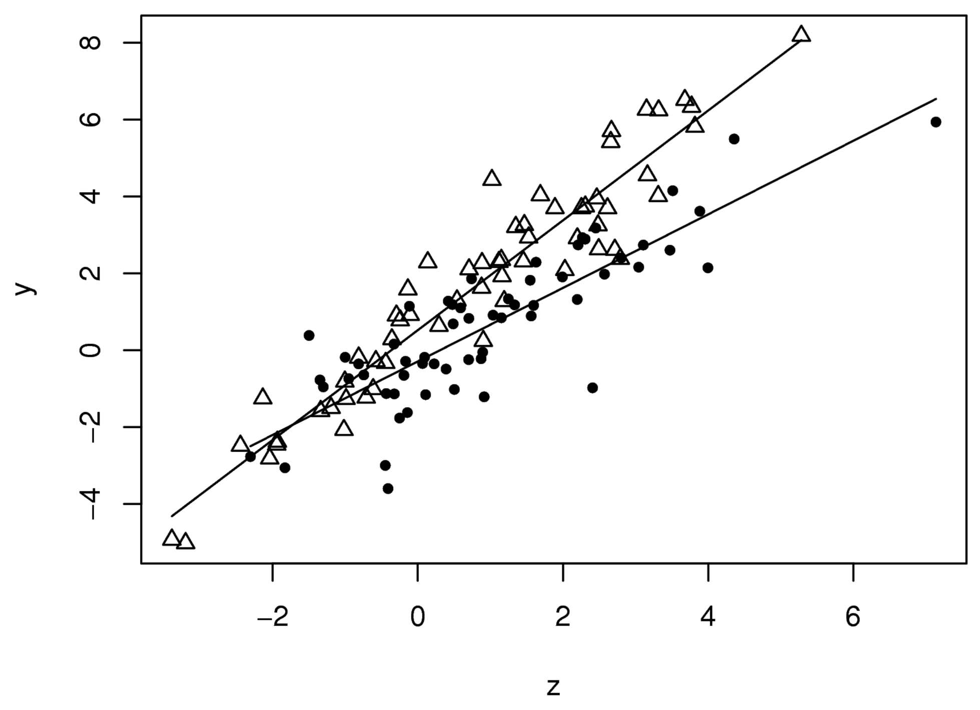

Figure 1: *Simulated data and model. The first half of observations is represented by circles ($y_t = z_t + e_t$), the second half by triangles ($y_t = 0.5 + 1.5 z_t + e_t$).*

References

[1] Bai J. and Perron P. (1998), Econometrica **66**, 47 – 78.

[2] Bai J. and Perron P. (2003), The Econometrics Journal **6**, 72 – 78.

[3] Hušková M. (2004), Asymptotic Methods in Stochastics, Vol. 44, 273 – 291.

[4] Marušiaková M. (2005), Master's thesis, Charles University, Prague.

[5] R Development Core Team (2005), ISBN 3-900051-07-0, URL http://www.R-project.org.

[6] Zeileis A., Leisch F., Hornik K. and Kleiber C. (2002), Journal of Statistical Software **7**, 1 – 38.

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				Normal errors				Laplace errors			
m	δ_1'	δ_2'	δ_3'	0.10	0.05	0.025	0.01	0.10	0.05	0.025	0.01
0	(0,1)	(0,1)	(0,1)	8.49	9.61	10.67	12.11	8.43	9.61	10.83	12.67
1	(0,1)	($\frac{1}{2}$,1)	($\frac{1}{2}$,1)	8.53	9.77	10.85	12.24	8.71	9.94	11.06	12.60
1	(0,1)	(1,1)	(1,1)	8.47	9.68	10.93	12.33	8.44	9.73	10.86	12.27
1	(0,1)	(0,$\frac{3}{2}$)	(0,$\frac{3}{2}$)	8.47	9.68	10.99	12.25	8.69	9.86	10.95	12.62
1	(0,1)	(0,2)	(0,2)	8.59	9.79	11.04	12.46	8.48	9.68	10.77	12.43
1	(0,1)	($\frac{1}{2}$,$\frac{3}{2}$)	($\frac{1}{2}$,$\frac{3}{2}$)	8.44	9.68	10.79	12.40	8.30	9.44	10.49	11.86
1	(0,1)	(1,2)	(1,2)	8.47	9.65	10.74	12.20	8.42	9.59	10.80	12.23
2	(0,1)	($\frac{1}{2}$,1)	(1,1)	8.65	9.74	10.96	12.62	8.73	10.07	11.48	12.97
2	(0,1)	(1,1)	(2,1)	8.47	9.73	10.77	12.09	8.62	9.73	10.94	12.44
2	(0,1)	(0,$\frac{3}{2}$)	(0,2)	8.49	9.63	10.72	12.20	8.58	9.82	11.08	12.57
2	(0,1)	(0,2)	(0,3)	8.60	9.79	11.14	12.67	8.56	9.76	10.88	12.29
2	(0,1)	($\frac{1}{2}$,$\frac{3}{2}$)	(1,2)	8.55	9.70	10.88	12.10	8.54	9.74	10.86	12.28
2	(0,1)	(1,$\frac{3}{2}$)	(2,2)	8.47	9.60	10.70	12.33	8.54	9.82	11.16	12.67
2	(0,1)	($\frac{1}{2}$,$\frac{1}{2}$)	(1,1)	8.47	9.62	10.77	12.29	8.56	9.72	10.92	12.32
2	(0,1)	(1,$\frac{3}{2}$)	(1,2)	8.60	9.70	10.89	12.54	8.48	9.74	10.92	12.31
BP ACV				8.63	9.75	10.75	12.15	8.63	9.75	10.75	12.15

Table 1: *Approximations to critical values of the test (2) for $k = 2, q = 2, \varepsilon = 0.15$. The entries are quantiles x such that $P(\sup F_n^{\varepsilon}(k; q) \leq x/q) = 1 - \alpha$ ($\alpha = 0.10, \dots, 0.01$). The asymptotic critical values (calculated in [1, 2]) are in the last row.*