ARBITRAGE OPPORTUNITY AND MARTINGALE PRICING MEASURES

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ABSTRACT. King and Korf [3] introduced a new framework for analyzing pricing theory for incomplete markets and contingent claims. The fundamental theorem of asset pricing was reformulated in a very general form. It claims that under the assumption of the essentially arbitrage-free market, the fair price of a contingent claim can be stated as a supremum of the expectation over an infinite set of equivalent finitely additive martingale probabilistic measures. We propose an equivalent characterization of the arbitrage-free market in the sense of no free lunch in the limit.

Резюме. Кинг и Корф [3] представили новую концепцию анализа теории оценения для неполного рынка и финансовых потоках. Они переформулировали фундаментальную теорему о оценении актив в общей форме. Эта теорема устанавливает, что если рынок без арбитража, то справедливая цена финансового потока есть супремум ожидания через множество эквивалентных конечно-аддитивных мартингаловых мер. Мы предлагаем характеризацию рынка без арбитража.

1. INTRODUCTION

King and Korf [3] introduced a new framework for analyzing pricing theory for incomplete markets and contingent claims. They used conjugate duality and stochastic optimization theory applied on the duality scheme $L^\infty/(L^\infty)^*$. For the history of applying duality in stochastic programming on infinite dimensional spaces we refer the reader to [3].

Various statements in the literature of the fundamental theorem of asset pricing give conditions under which an arbitrage-free market is equivalent to the existence of an equivalent martingale measure. A formula for the fair price of a replicated contingent claim is given as an expectation with respect to such a measure. In the setting of incomplete markets, the fair price is a supremum over a set of equivalent martingale measures.

In [3], the fundamental theorem of asset pricing was reformulated in the very general form. It claims that under the assumption of the essentially arbitrage-free market, the fair price of a contingent claim can be stated as a supremum of the expectation over an infinite set of equivalent finitely additive martingale probabilistic measures. An arbitrage opportunity in the market is characterized by a free lunch in the limit that is slightly weaker than a usual definition of free lunch.

In Sections 2-4, the mathematical overview of the financial terminology is given, the writer's problems are specified, and no free lunch in the limit is introduced. In

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Section 5 we propose an equivalent characterization of the arbitrage-free market in the sense of no free lunch in the limit.

2. Model specification

We begin with a mathematical overview of the necessary financial terminology (for more details see [3]). The underlying *market* is a collection of $J+1$ traded assets indexed by $j = 0, \dots, J$. Each asset has an initial *market price* at time $t = 0$, and future market prices at times $t = 1, \dots, T$. The prices are described by a nonnegative vector $S_0 = (S_0^0, \dots, S_0^J)^\top \in \mathbb{R}_+^{J+1}$ of future market prices, where $(\Omega, \mathcal{F}, P)$ is an underlying probability space with P-complete σ-algebra $\mathcal{F}$ generated by a filtration $\mathcal{F}_t$ with $\mathcal{F}_T = \mathcal{F}$. It is assumed that the first asset is risk-free in the sense that its market price is always strictly positive ($S_t^0 > 0, t = 0, \dots, T$). This asset is the *numeraire*. Using it to normalize the values of all other assets, we can get the new *discounted price* vectors $Z_t = S_t/S_t^0$. It is assumed that all other prices and cash flows have been similarly adjusted to reflect this normalization. Prices in the price vector Z_t are assumed to be $\mathcal{F}_t$-measurable and essentially bounded. Let us denote by $L^\infty(\Omega, \mathcal{F}, P; A)$ the set of all $\mathcal{F}$-measurable functions w on Ω such that $|w(\omega)| \leq M$ P-almost surely for some M (we follow the notation of [4]). Hence assume all variables $Z_t \in L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}_+^{J+1})$.

An *investor* may hold a *portfolio* of assets $j = 0, \dots, J$, described by a vector $\theta_t = (\theta_t^0, \dots, \theta_t^J)^\top, t = 0, \dots, T$. The investor has some initial wealth to invest, and may change his or her portfolio at each time $t = 0, \dots, T$. The decision of the portfolio arrangement will depend on the market behaviour. A *trading strategy* describes all investment decisions based on all possible outcomes of the market. Therefore, $\theta = (\theta_0, \dots, \theta_T)$ describes a trading strategy, where at time $t = 0$, the market prices are known and θ_0 is described by a vector in $\mathbb{R}^{J+1}$. At time $t = 1, \dots, T$, the market prices are $\mathcal{F}_t$-measurable functions on Ω, so that $\theta_t : \Omega \to \mathbb{R}^{J+1}$ is also $\mathcal{F}_t$-measurable, and describes the portfolio at time t. Note that θ_t is allowed to take on negative values, which corresponds to borrowing. The class of all possible strategies is limited to those which are essentially bounded.

A *self-financing trading strategy* is one in which no new money is required or generated to create it. This is expressed by $Z_t^\top \theta_t = Z_t^\top \theta_{t-1}$ P-a.s. for all $t = 1, \dots, T$. It is convenient to adopt the notation $\Delta\theta_t = \theta_t - \theta_{t-1}$. Obviously, $\Delta\theta_t$ is $\mathcal{F}_t$-measurable.

Next we define a *contingent claim*. A contingent claim is a type of contract that is contingent on the underlying market. Precisely, in our setting, it is a promise to pay $F_t : \Omega \to \mathbb{R}$ at each time $t = 1, \dots, T$, where F_t is $\mathcal{F}_t$-measurable. We assume again that F_t is $\mathcal{F}_t$-measurable and essentially bounded. It could take negative values.

3. The writer's problems

The *writer* of a contingent claim will price the claim at a *fair* price in consideration of the fact that he will be able to invest his earnings from the sale in the market. Assuming for now that this price has been fixed at F_0, the *writer's portfolio optimization problem* (P_w) is given by

$$
\begin{array}{ll}
\max & E[Z_T^\top \theta_T] \\
\text{subject to} & Z_0^\top \theta_0 \leq F_0 \\
& Z_t^\top \Delta\theta_t \leq -F_t \quad P\text{-a.s.}\,,\ t = 1, \dots, T \\
& Z_T^\top \theta_T \geq 0 \quad P\text{-a.s.}
\end{array}
\qquad (P_w)
$$

Associated with the problem (P_w) is the *writer's pricing problem* (P_{wp}) in which the writer determines the fair price of the contingent claim as the lowest price F_0 such that (P_w) is feasible:

$$
\begin{array}{ll}
\min & V \\
\text{subject to} & Z_0^\top \theta_0 - V \le 0 \\
 & Z_t^\top \Delta\theta_t \le -F_t \quad P\text{-a.s. },\ t = 1,\dots,T \\
(2) & Z_T^\top \theta_T \ge 0 \quad P\text{-a.s.}
\end{array}
\qquad (P_{wp})
$$

These two problems can be stated in a general form using the writer's utility function $u : \mathbb{R} \to \overline{\mathbb{R}}$ as a problem (P_u)

$$
\begin{array}{ll}
\max & E[u(Z_T^\top \theta_T)] \\
\text{subject to} & Z_0^\top \theta_0 \le F_0 \\
(3) & Z_t^\top \Delta\theta_t \le -F_t \quad P\text{-a.s. },\ t = 1,\dots,T
\end{array}
\qquad (P_u)
$$

and (P_{up}), respectively:

$$
\begin{array}{ll}
\min & V \\
\text{subject to} & Z_0^\top \theta_0 - V \le 0 \\
 & Z_t^\top \Delta\theta_t \le -F_t \quad P\text{-a.s. },\ t = 1,\dots,T \\
 & Z_T^\top \theta_T \in \operatorname{cl}\operatorname{dom} u \quad P\text{-a.s.}
\end{array}
\qquad (P_{up})
$$

We assume that the utility function u is concave, strictly increasing, and upper semi-continuous, with $u(x) \to \infty$ as $x \to \infty$. In the particular instances (P_w) and (P_{wp}), the utility function takes the form

$$
u(x) = \begin{cases} x, & x \ge 0 \\ -\infty, & x < 0 \end{cases}
$$

In [3], the dual problems were formulated using theory of conjugate duality and optimization in the $L^\infty/(L^\infty)^*$ stochastic programming duality scheme (see [5]). For the definition and basic properties of L^∞ and $(L^\infty)^*$ see [2].

Especially note that each element $\bar{y} \in (L^\infty)^*$ may be uniquely decomposed into an L^1 component y and a singular component y^0. An element $y^0 \in (L^\infty)^*$ is singular if there exists a sequence of sets $E^\nu \subset \Omega$ with $P(E^\nu) \searrow 0$ such that for all $z \in L^\infty$ if $z\mathbf{1}_{E^\nu} = 0$ P-a.s. for some ν, then $y^0(z) = 0$.

The problem dual to (P_u) is

$$
\begin{array}{ll}
\min & F_0 y_0 - \sum_{t=1}^{T} E[F_t y_t] - \sum_{t=1}^{T} \langle y_t^0, F_t \rangle - (Eu)^*(y_T, y_T^0) \\
\text{subject to} & E[y_t Z_t^\top \theta_{t-1}] + \langle y_t^0, Z_t^\top \theta_{t-1} \rangle = E[y_{t-1} Z_{t-1}^\top \theta_{t-1}] + \langle y_{t-1}^0, Z_{t-1}^\top \theta_{t-1} \rangle \\
\text{for all} & \theta_{t-1} \in L^\infty(\Omega, \mathcal{F}_{t-1}, P; \mathbb{R}^{J+1}),\ t = 1,\dots,T \\
 & y \ge 0 \\
(5) & y \in Y
\end{array}
\qquad (D_u)
$$

and the problem dual to (P_{up}) is

$$
\begin{array}{rll}
\max & \sum_{t=1}^{T} E[F_t y_t] + \sum_{t=1}^{T} \langle y_t^0, F_t\rangle + E[\alpha x_T] + \langle x_T^0, \alpha \mathbf{1}\rangle & \\
\text{subject to} & E[y_t Z_t^\top \theta_{t-1}] + \langle y_t^0, Z_t^\top \theta_{t-1}\rangle = E[y_{t-1} Z_{t-1}^\top \theta_{t-1}] + \langle y_{t-1}^0, Z_{t-1}^\top \theta_{t-1}\rangle & \\
\text{for all} & \theta_{t-1} \in L^\infty(\Omega, \mathcal{F}_{t-1}, P; \mathbb{R}^{J+1}),\ t = 1, \dots, T & \\
& x_T^0 = y_T^0,\ x_T = y_T\ P\text{-a.s.}\ ,\ y_0 = 1 & \\
& y \geq 0 & (D_{up}) \\
(6) & y \in Y &
\end{array}
$$

where $\langle .,. \rangle$ denotes the bilinear form on $(L^\infty)^* \times L^\infty$ (in fact, $\langle y, z\rangle = y(z)$),

$$Y = \{y = (y_0, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_T) : y_0 \in \mathbb{R}, \bar{y}_t = (y_t, y_t^0) \in (L^\infty)^*(\Omega, \mathcal{F}_t, P; \mathbb{R})\},$$

$t = 1, \dots, T$, with the weak* product topology, and
$y \geq 0$ with $y \in Y$ means that $y_0 \geq 0, y_t \geq 0$ P-a.s. and $\langle y_t^0, z\rangle \geq 0$ for all $z \in L_+^\infty(\Omega, \mathcal{F}_t, P; \mathbb{R}), t = 1, \dots, T$.
The functional $Eu : L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}) \to \overline{\mathbb{R}}$ is defined by $Eu(x) = E[u(x)]$,
$(Eu)^* : (L^\infty)^*(\Omega, \mathcal{F}, P; \mathbb{R}) \to \overline{\mathbb{R}}$ is the conjugate of Eu in the concave sense, which means

$$(Eu)^*(y) = \inf_{w \in L^\infty(\Omega, \mathcal{F}, P; \mathbb{R})} \{\langle y, w\rangle - Eu(w)\},$$

$\alpha = \inf \operatorname{dom} u > -\infty$, and $\mathbf{1} = 1$ P-a.s.

Recall that the problem (P_u) is said to be *strictly feasible* if there exists $\varepsilon > 0$, $\theta \in \Theta$ such that

$$
\begin{array}{rcl}
Z_0^\top \theta_0 & \leq & F_0 - \varepsilon \\
Z_t^\top \Delta\theta_t & \leq & -F_t - \varepsilon \quad P\text{-a.s.}\ ,\ t = 1, \dots, T.
\end{array}
$$

There is a duality theorem relating these problems.

Theorem 3.1. *Suppose (P_u) is strictly feasible. Then* $\sup(P_u) = \min(D_u)$.

4. No free lunch in the limit

The *arbitrage* in the market means that there is a possibility to generate positive wealth with no risk. The market is said to admit *no free lunch* if there are no self-financing trading strategies with zero initial wealth, nonnegative terminal wealth, and with a positive probability of strictly positive terminal wealth. In [3], a new concept called *no free lunch in the limit* is introduced. It means that there is no sequence of trading strategies satisfying

$$
\begin{array}{rcl}
Z_0^\top \theta_0^\nu & = & 0 \\
Z_t^\top \Delta\theta_t^\nu & = & 0 \quad P\text{-a.s.}\ , t = 1, \dots, T \\
Z_T^\top \theta_T^\nu & \geq & -\varepsilon^\nu \quad P\text{-a.s.} \\
\lim_{\nu\to\infty} E[Z_T^\top \theta_T^\nu] & > & 0
\end{array}
$$

where $\varepsilon^\nu \searrow 0$. The next theorem equates no free lunch in the limit with the boundedness of the problem (P_w).

Theorem 4.1. *Suppose (P_w) is strictly feasible with $F_0 > \operatorname{ess\,inf}(\sum_{t=1}^T F_t)$. Then the following are equivalent.*

(i) *(P_w) is bounded,*
(ii) *the market admits no free lunch in the limit.*

5. The fundamental theorem of asset pricing

In [3], the pricing theory for contingent claims in incomplete market is derived. The fair price is taken as inf (P_{up}).

Recall that $\{Z_t\}_{t=0}^T$ is a *martingale* under a probabilistic measure Q if

$$E[Z_t|\mathcal{F}_{t-1}] = Z_{t-1}\ Q\text{-a.s.},\ t = 1, \dots, T.$$

If in addition $Q \ll P$, we call Q the *martingale measure* for the process $\{Z_t\}_{t=0}^T$.

Theorem 5.1. *Suppose the market admits no free lunch in the limit. Then the writer's fair price is*

$$\max_{Q \in \mathcal{Q}} \left\{ \sum_{t=1}^{T} E_Q[F_t] + \alpha \right\}$$

where $\mathcal{Q}$ denotes the space of finitely additive martingale measures and $\alpha = \inf \operatorname{dom} u > -\infty$.

Also the equivalent characterization of the arbitrage-free market is discussed. But Lemma 7.1 and consequently Theorem 7.2 in [3] should be formulated more carefully. Let us present the corrected version of the theorem and its proof.

Lemma 5.2. *Problem (D_u) is feasible if and only if there exists a finitely additive equivalent martingale measure Q on Ω such that $\inf_{w \in L^\infty(\Omega,\mathcal{F},P;\mathbb{R})}\{E_Q[w] - E_P[u(w)]\} > -\infty$.*

Proof. Let $y = (y_0, (y_1, y_1^0), \dots, (y_T, y_T^0)) \in Y$ be feasible for (D_u). Then y satisfies the constraints in (D_u)

$$\begin{aligned} &E[y_t Z_t^\top \theta_{t-1}] + \langle y_t^0, Z_t^\top \theta_{t-1}\rangle = E[y_{t-1} Z_{t-1}^\top \theta_{t-1}] + \langle y_{t-1}^0, Z_{t-1}^\top \theta_{t-1}\rangle \\ &\qquad \text{for all } \theta_{t-1} \in L^\infty(\Omega, \mathcal{F}_{t-1}, P; \mathbb{R}^{J+1}),\ t = 1, \dots, T, \\ &(y_T, y_T^0) \in \operatorname{dom}(Eu)^*,\ y \geq 0. \end{aligned}$$

It can be shown that $y_T > 0\ \ P$-a.s. as it was in [3].

Let $\bar{v}_T = (v_T, v_T^0)$ where

$$v_T = \frac{y_T}{E[y_T] + \langle y_T^0, \mathbf{1}\rangle}$$

and

$$v_T^0 = \frac{y_T^0}{E[y_T] + \langle y_T^0, \mathbf{1}\rangle}.$$

We show that the set function Q defined by

$$Q(E) = \langle \bar{v}_T, \mathbf{1}_E\rangle, E \in \mathcal{F}$$

is an equivalent finitely additive martingale measure. The finite additivity is induced by such property of (y_T, y_T^0). By normalization, $Q(\Omega) = \langle \bar{v}_T, \mathbf{1}\rangle = \frac{E[y_T]}{E[y_T]+\langle y_T^0,\mathbf{1}\rangle} + \frac{\langle y_T^0,\mathbf{1}\rangle}{E[y_T]+\langle y_T^0,\mathbf{1}\rangle} = 1$. Since $v_T > 0$, Q is equivalent to P. Now we show that Q is a martingale measure for the price process Z. Let $v_t = E[v_T|\mathcal{F}_t]$ and v_t^0 be the unique singular component which is $\mathcal{F}_t$-measurable and satisfies

$$\langle v_t^0, \mathbf{1}_E\rangle + E[v_T \mathbf{1}_E] = Q(E) \text{ for all } E \in \mathcal{F}_t, t = 0, \dots, T.$$

Then

$$v_t = \frac{y_t}{E[y_T] + \langle y_T^0, \mathbf{1}\rangle}$$

and

$$v_t^0 = \frac{y_t^0}{E[y_T] + \langle y_T^0, \mathbf{1}\rangle}.$$

Thus by the constraints in (D_u), we obtain

$$\begin{aligned} E_Q[Z_t^\top \theta_{t-1}] &= E[v_T Z_t^\top \theta_{t-1}] + \langle v_T^0, Z_t^\top \theta_{t-1}\rangle \\ &= E[v_t Z_t^\top \theta_{t-1}] + \langle v_t^0, Z_t^\top \theta_{t-1}\rangle \\ &= E[v_{t-1} Z_{t-1}^\top \theta_{t-1}] + \langle v_{t-1}^0, Z_{t-1}^\top \theta_{t-1}\rangle \\ &= E_Q[Z_{t-1}^\top \theta_{t-1}]. \end{aligned}$$

Hence Q is an equivalent finitely additive martingale measure. Since $(y_T, y_T^0) \in \mathrm{dom}(Eu)^*$, we have

$$\inf_{w\in L^\infty(\Omega,\mathcal{F},P;\mathbb{R})}\{E_Q[w]-E[u(w)]\} = \inf_{w\in L^\infty(\Omega,\mathcal{F},P;\mathbb{R})}\{E[wy_T]+\langle y_T^0, w\rangle - E[u(w)]\} > -\infty$$

Now suppose that there exists an equivalent finitely additive martingale measure Q such that $\inf_{w\in L^\infty(\Omega,\mathcal{F},P;\mathbb{R})}\{E_Q[w]-E[u(w)]\} > -\infty$. Let (y, y^0) be the representation of Q in $(L^\infty)^*(\Omega,\mathcal{F},P;\mathbb{R})$. By assumption, we get

$$(Eu)^*(y, y^0) = \inf_{w\in L^\infty(\Omega,\mathcal{F},P;\mathbb{R})}\{E_Q[w]-E[u(w)]\} > -\infty,$$

and hence $(y, y^0) \in \mathrm{dom}(Eu)^*$.

Let $y_t = E[y|\mathcal{F}_t]$ and let y_t^0 be the unique $\mathcal{F}_t$-measurable singular component such that

$$\langle y_t^0, \mathbf{1}_E\rangle + E[y\mathbf{1}_E] = Q(E) \text{ for all } E \in \mathcal{F}_t, t = 0, \dots, T.$$

Then $y = (y_0, (y_1, y_1^0), \dots, (y_T, y_T^0)) \in Y$ is a feasible solution to (D_u), which completes the proof. □

Corollary 5.3. *Problem (D_w) is feasible if and only if P is the martingale measure for the price process.*

Proof. It is sufficient to make the following observation. Let us assume that there exists a finitely additive equivalent martingale measure Q on Ω such that $\inf_{w\in L_+^\infty(\Omega,\mathcal{F},P;\mathbb{R})}\{E_Q[w] - E_P[w]\} > -\infty$. We show that in such case $Q = P$. On the contrary, suppose that there exists $E \in \mathcal{F}$ such that $Q(E) \neq P(E)$. Without loss of generality we assume $P(E) > Q(E)$. Let $w^\nu = \nu\mathbf{1}_E$. Then $E_Q[w^\nu] - E_P[w^\nu] = \nu(Q(E) - P(E)) \searrow -\infty$ as $\nu \to \infty$, which contradicts the assumption. □

Theorem 5.4. *The market admits no free lunch in the limit if and only if P is the martingale measure for the price process.*

Proof. Consider the writer's portfolio optimization problem (P_w) with (P_w) strictly feasible and satisfying $F_0 > \operatorname{ess\,inf} \sum_{t=1}^T F_T$. By Theorem 4.1, no free lunch in the limit is equivalent to the boundedness of (P_w). This is equivalent to the feasibility of (D_w) through the duality result in Theorem 3.1. Finally, Corollary 5.3 gives us the equivalence between the feasibility of (D_w) and the property of P being the martingale measure. □

In fact, the original concept of free lunch does not depend on the choice of the equivalent underlying probabilistic measure P, but the free lunch in the limit does. Therefore in Theorem 5.4, we cannot talk about any equivalent martingale measure, but just the underlying probabilistic measure P. There is no general method to specify whether or not a probabilistic martingale measure for the price process exists. The answer is known in the case of $\mathcal{F}$ finite (see e.g. [1]).

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