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STATISTICAL MODELS AND ANALYSIS OF CUMULATED DAMAGE PROCESSES

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ABSTRACT. The contribution deals with a stochastic process which cumulates random increments at random moments. It is described by the intensity of random (counting) process of these moments and by a distribution of increments. The resulting process is here called the cumulative process. We derive its martingale - compensator decomposition and then we propose the estimator of characteristics of distribution of increments. An application deals with the process of growing damage of a technical device leading to a break of device when the cumulated damage exceeds a certain level. We also discuss the problem of prediction of the cumulative process behaviour.
V článku se zabýváme procesem kumulujícím náhodné přírůstky v časech daných náhodným bodovým procesem. Uvažujeme poměrně obecný model s martingalovou strukturou "inovace" procesu, tj. proces rozložitelný na martingal a kompenzátor adaptovaný na příslušnou filtraci. Uvažujeme model jako regresní a zabýváme se odhadem intenzity bodového procesu a charakteristik rozdělení přírůstků. Je také zkoumán problém predikce chování procesu. Jako aplikaci analyzujeme proces růstu poškození výrobku či materiálu.
Резюмэ: В статье рассмотрены случайные точечные и кумулативные процессы – случайные суммы. Показано разложение процесса в мартингал и компензатор. Далее представлены оценьки интенсивности и характеристик приращений, обсуждается вопрос предсказания поведения процесса. В качестве примера исследованы процессы накопления повреждения, которые влияют на вероятность отказа технического продукта.

1. THE MODEL OF CUMULATIVE PROCESS

The counting process is a stochastic point process registering random events and counting their number. The trajectory of such a process starts at zero and has jumps +1 at random moments of events. The main characteristic is the intensity of the stream of events. A review of theory and applications of counting process models is given, for instance, in Andersen et al (1993), or Fleming and Harrington (1991).

In the present paper, we consider random process

$$C(t) = \int_0^t Y(s)\,\mathrm{d}N(s), \qquad (\ C(0) = 0\ ), \tag{1}$$

where $N(t)$ is a counting process and $Y(t)$ is a set of random variables. Such a process combining the counting process with the process of random increments is called the compound counting process, or sometimes also the cumulative process

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(Volf, 2000). Its model is suitable for the description of many real-world technological, environmental, biological and also financial processes (especially in the field of insurance, cf. Embrechts et al, 1997, Rolski et al, 1999). The objective of the present paper is to describe the process (1) with the aid of characteristics of both its components, i. e. the hazard function of $N(t)$ and the distribution of $Y(t)$, and to apply it to the modeling of growing damage of a technical device. We also discuss the problem of prediction of the cumulative process behaviour.

In the scenario considered in Volf (2000) it was assumed that each $Y(t)$ was independent of the history of the process $C(s)$ up to t (on the other hand, the intensity of $N(t)$ could depend on the history). It is a rather strong condition which in some cases is not fulfilled, though we can imagine a number of examples for which such an independence of increments on the history is a quite realistic property.

In the present paper we propose a model allowing for the dependence of distribution of $Y(t)$ on $\mathcal{S}(t^-)$, where $\mathcal{S}(t)$ is a corresponding filtration, i.e. a nondecreasing sequence of σ-algebras defined on the sample space of $\{N(s), J(s), \boldsymbol{Z}(s), Y(s), 0 \le s \le t\}$, so that $\mathcal{S}(t^-)$ is its left-continuous version, a 'history'. Intensity of $N(t)$ is $\lambda(t) = h(t, Z(t))J(t)$, cumulative intensity $L(t) = \int_0^t \lambda(s)\mathrm{d}s$, $h(t,z)$ is a hazard function, $J(t)$ and $\boldsymbol{Z}(t)$ are $\mathcal{S}(t^-)$ measurable predictable processes, e.g. an indicator of observability of $C(t)$ ($J(t) = 1$ if $C(t)$ is observed, $J(t) = 0$ otherwise), and a covariate process. It is assumed that they are left-continuous, while the processes $C(t)$ and $N(t)$ are right-continuous.

As regards the distribution of random variables $Y(t)$, we assume that the conditional distribution of $Y(t)$, given $\mathcal{S}(t^-)$, can be described via a density function $f(y; t, \boldsymbol{Z}(t))$, and it possesses the first and second conditional moments $E(Y(t)|S(t^-)) = \mu(t, \boldsymbol{Z}(t))$, $var(Y(t)|S(t^-)) = \sigma^2(t, \boldsymbol{Z}(t))$. These definitions imply that the processes $N(t), C(t)$ depend on $\mathcal{S}(t^-)$ through $\boldsymbol{Z}(t)$ and $J(t)$.

The processes are followed throughout a time interval $[0, T]$, the covariate can be multidimensional and its values are from a set $\mathcal{Z} \in R^K$, say. For the sake of better estimability, let us assume that functions h, μ, σ are bounded and continuous. By the way, if $h(t, z)$ is a bounded function, it follows that the probability of two events at one moment is zero, which is one of basic assumptions of the event-history models based on continuous–time counting processes.

A similar case, with parametrized function μ, has been studied in Scheike (1994), with the focus on estimation of the parameter, and, eventually, on a kernel estimation of both μ, σ in a nonparametrized scheme (cf. Orsáková in the present volume, and references in Scheike, 1994). Our objective is to derive the estimator of the rate of the cumulative process and of the characteristics of random increments. Then the model will be applied to the description of processes of growing damage, with an numerical example and with an attempt to predict the future behaviour of process.

1.1. **The compensator of cumulative process.** Let us now recall the compensator – martingale decomposition of the counting process, namely $N(t) = L(t) + M(t)$, $M(t)$ being the martingale adapted to σ-algebras $\mathcal{S}(t)$, with variance process $L(t)$ (cf. for instance Andersen et al 1993). Notice also that under our assumptions, $Y(t)$ is conditionally independent, given $\mathcal{S}(t^-)$, of $\mathrm{d}M(t)$, where by $\mathrm{d}M(t)$ we denote the increment of $M(t)$ in a small interval $[t, t + \mathrm{d}t)$. Similarly, by $\mathrm{d}\langle M\rangle(t) = var\{\mathrm{d}M(t)|\mathcal{S}(t^-)\}$ we mean the increment of the variance process of $M(t)$, which we denote by $\langle M\rangle(t)$. This martingale innovation structure is crucial for the theory and methods of statistical inference, namely for the consistency and asymptotic normality of estimates. That is why we search for a similar decomposition of the cumulative process, too.

Let us denote $Y^*(t) = Y(t) - \mu(t, \boldsymbol{Z}(t))$, so that $E(Y^*(t)|\mathcal{S}(t^-)) = 0$. Then we can write

$$C(t) = \int_0^t (Y^*(s) + \mu(s, \boldsymbol{Z}(s)))\, \mathrm{d}N(s) = \int_0^t \mu(s, \boldsymbol{Z}(s))\, \mathrm{d}L(s) + \mathcal{M}(t),$$

where

$$\mathcal{M}(t) = \mathcal{M}_1(t) + \mathcal{M}_2(t) = \int_0^t Y^*(s)\, \mathrm{d}N(s) + \int_0^t \mu(s, \boldsymbol{Z}(s))\, \mathrm{d}M(s).$$

Proposition 1. The processes $\mathcal{M}(t),\ \mathcal{M}_1(t),\ \mathcal{M}_2(t)$ are martingales adapted to σ-algebras $\mathcal{S}(t)$, the variance process of $\mathcal{M}(t)$ is

$$\langle \mathcal{M} \rangle(t) = \int_0^t (\sigma^2(s, \boldsymbol{Z}(s)) + \mu^2(s, \boldsymbol{Z}(s)))\, \mathrm{d}L(s).$$

The proposition is proved in Volf (2000).

Corollary. Process $\int_0^t \mu(s, \boldsymbol{Z}(s))\, \mathrm{d}L(s)$ is the compensator of process $C(t)$.

Evidently, process $\int_0^t \mu(s, \boldsymbol{Z}(s))\, \mathrm{d}L(s)$ is $\mathcal{S}(t^-)$-measurable and predictable (the paths are continuous).

2. Large sample properties

Let n realizations $C_i(t) = \int_0^t Y_i(s)\, \mathrm{d}N_i(s)$ of process $C(t)$, together with corresponding processes $J_i(t), \boldsymbol{Z}_i(t)$, be observed in an interval of the interest, $[0, T]$. More precisely, we observe the paths of processes $J_i(t), \boldsymbol{Z}_i(t)$, and, provided $J(t) = 1$, the moments of events T_{ij} of counting processes $N_i(t)$ and increments $Y_i(T_{ij})$ (for $i = 1, \ldots, n,\ j = 1, \ldots, n_i = N_i(T)$). It is assumed that random variables $Y_i(t), i = 1, 2, ..., n$ have the same conditional probability densities $f(y; t, \boldsymbol{z})$ and that $N_i(t)$ are characterized by the same hazard function $h(t, z)$. Now the common filtration $\mathcal{S}(t)$ is constructed above all paths of $\{C_i(s), N_i(s), J_i(s), \boldsymbol{Z}_i(s), s \le t, i = 1, 2, ..., n\}$. Counting processes $N_i(t)$ have intensities $\lambda_i(t) = h(t)J_i(t)$, by $L_i(t) = \int_0^t \lambda_i(s)\mathrm{d}s$ we denote the cumulative intensity processes, $M_i(t) = N_i(t) - L_i(t)$ are martingales. As we assumed the boundedness of $h(t, z)$, and also processes $J_i(t)$ are bounded, than the probability of two increments at one moment is zero and the martingales are mutually orthogonal, i.e. $\mathrm{d}\langle M_i, M_j \rangle(t) = 0$ for $i \ne j$. The same holds for $\mathcal{M}_i(t)$, the martingales defined in the same way as $\mathcal{M}(t)$ in the preceding section, i.e. $\mathrm{d}\langle \mathcal{M}_i, \mathcal{M}_j \rangle(t) = 0$ for $i \ne j$. Actually, from this impossibility of simultaneous events it also follows that the increments of $C_i(t)$ are conditionally orthogonal, given the history of the process.

From the conditional independence of innovation of the processes the multiplicative form of the likelihood process follows (it is actually a generalization of the likelihood function of Poisson process):

$$\mathcal{L} = \prod_{i=1}^{n} \prod_{j=1}^{n_i} \{\lambda_i(T_{ij}) f(Y_i(T_{ij}); T_{ij}, \boldsymbol{Z}_i(T_{ij}))\} \cdot \exp\left\{ -\int_0^T \lambda_i(t)\, \mathrm{d}t \right\},$$

where $\lambda_i(t) = h(t, \boldsymbol{Z}_i(t))J_i(t)$. Consequently, the part containing the intensities and the part containing the distribution of Y's are separated (and therefore both characteristics can be estimated independently). In the case of parametrized function f, its parameters can be estimated from the maximum likelihood estimation procedure based on $\mathcal{L}(f) = \prod_{i=1}^{n} \prod_{j=1}^{n_i} f(Y_i(T_{ij}); T_{ij}, \boldsymbol{Z}_i(T_{ij}))$ only. In a nonparametrized case, estimates of functions $\mu(t,z)$, $\sigma^2(t,z)$ can be obtained with the help of the smoothing (kernel) technique. Even the density $f(y;t,z)$ could be then estimated via the kernel method.

2.1. **Estimates and their convergence.** Let us first recall several results from Volf (2000), where the influence of covariates was not considered. In such a case, the most common estimator of the cumulative hazard function $H(t) = \int_0^t h(s)ds$ is the Nelson-Aalen one:

$$\hat{H}_n(t) = \sum_{i=1}^{n} \int_0^t \frac{1[J(s) > 0]}{J(s)} \mathrm{d}N_i(s) = \int_0^t \frac{1[J(s) > 0]}{J(s)} \mathrm{d}N(s), \tag{2}$$

where $J(s) = \sum_{i=1}^{n} J_i(s), N(s) = \sum_{i=1}^{n} N_i(s)$ (and we put $0/0 = 0$ when $J(s) = 0$). It is well known that such an estimator is uniformly consistent and asymptotically normal (in the sense of the weak convergence of normalized residual process to a Wiener process) on $[0,T]$, provided $J(s)$ tends to infinity uniformly in the whole interval. Let us assume even a stronger condition:

A1. There exists the limit $r(s) = \lim_{n\to\infty} \frac{J(s)}{n}$ in probability such that

a) the limit is uniform on $[0,T]$,

b) $r(s) \geq e$ on $[0,T]$, for some $e > 0$.

As an analogy to (2), let us now define the following "averaged" processes:

$$\bar{C}_n(t) = \sum_{i=1}^{n} \int_0^t \frac{1[J(s) > 0] Y_i(s)}{J(s)} \,\mathrm{d}N_i(s), \qquad \bar{K}(t) = \int_0^t \frac{\mu(s))}{r(s)} \,\mathrm{d}H(s),$$

where function $\bar{K}(t)$ actually represents an averaged rate of the development of the process $C(t)$. Under assumption **A1**, the following large sample results can be proven:

Proposition 2. $\bar{C}_n(t)$ is a uniformly consistent estimate of $\bar{K}(t)$ on $[0,T]$, i.e. $\lim_{n\to\infty} \sup_{t\in[0,T]} |\bar{C}_n(t) - \bar{K}(t)| = 0$ in probability.

If a proper version of Lyapunov condition is added, for instance that uniformly bounded $E\{|Y(t)|^3\}$ exists, asymptotic normality can be shown, too:

Proposition 3. The process $\sqrt{n}(\bar{C}_n(t) - \bar{K}(t))$ converges weakly on $[0,T]$ to a continuous Gaussian process with zero mean and independent increments, which has the variance function $w(t) = \int_0^t \frac{(\mu^2(s)+\sigma^2(s))}{r^2(s)} \mathrm{d}H(s)$.

On the basis of this convergence, the statistical tests, both of the goodness-of-fit and of homogeneity, were derived in Volf (2000). They used, similarly as the Kolmogorov-Smirnov type tests, the crossing probability results of Brownian motion and Brownian bridge processes. An application to the analysis of sequences of financial transactions, with particular attention to detection of atypical (possibly 'fraud') set of transactions, was suggested.

Let us now return to the more complex setting with regression on covariates (processes) $Z(t)$. Then the average from n cumulative processes is

$$\overline{C}_n(t) = \sum_i \int_0^t \frac{\varphi_i(s)}{J(s)} \,\mathrm{d}N_i(s) = \sum_i \int_0^t \left\{ \frac{\mu(s, Z_i(s))}{J(s)} \,\mathrm{d}L_i(s) + \frac{\mathrm{d}\mathcal{M}_i(s)}{J(s)}, \right\}$$

where the first part (a compensator) characterizes again an average (random) rate of growth of the process, while the second part tends to zero. It is possible to formulate a set of 'stability' assumptions, for instance requiring existence of limits

$\bar{\mu}(t) = lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mu(t, \boldsymbol{Z}_i(t))J_i(t)$, such that certain variants of Propositions 2 and 3 hold. However, the meaning of $\bar{\mu}(t)$ is then rather vague.

2.2. **Estimation of process' characteristics.** Another and more practical problem is the estimation of moments (and density function, say) of distribution of $Y(t)$ provided $Z(t) = z$. First, the distribution of the design of random points $(T_{ij},\, Z_i(T_{ij})),\ i = 1, \dots, n,\ j = 1, \dots, n_i$, should be estimated, then it will be used in the kernel estimation of "regression functions" $\mu(t, z),\ \sigma^2(t, z)$.

The basic results from the field of kernel estimation of density function and of regression function are summarized elsewhere, also in 'Robust' papers of Antoch (1982, 1986), Michálek (1994). Let us consider the simplest versions of estimates and their consistency at a given point (t, z) (and the case with random design of points $(T_{ij},\, Z_i(T_{ij}))$. Let $W_1(u)$ and $W_2(w)$ be two kernel functions. At point (t, z) let us define

$$\widehat{\varphi}(t, z) = \frac{1}{d_1 d_2 N(T)} \sum_{i=1}^{n} \int_0^T W_1\left(\frac{t-s}{d_1}\right) W_2\left(\frac{z - Z_i(s)}{d_2}\right) \mathrm{d}N_i(s),$$

where

$$N(T) = \sum_{i=1}^{n} \int_0^T \mathrm{d}N_i(t).$$

Further, assume that d_1 and $d_2 \to 0$, while $d_1 d_2 N(T) \to \infty$ when $n \to \infty$. Actually, points $(T_i, Z_i(T_i))$ are not i.i.d. variables, nevertheless, we just assume the following:
A2: $\widehat{\varphi}(t, z)$ converges (at given (t,z)) to a value $\varphi(t, z) > 0$, in probability,

where $\varphi(t, z)$ taken as a function on $[0, T] \times \mathcal{Z}$ means a density of marginal distribution of one (observed) point $(t, Z(t))$ for a set of processes $Z(t),\ J(t)$.

Let us define the following estimator at point (t, z):

$$\widehat{\mu}(t, z) = \frac{1}{\widehat{\varphi}(t, z) d_1 d_2 N(T)} \sum_{i=1}^{n} \int_0^T Y_i(t)\, W_1\left(\frac{t-s}{d_1}\right) W_2\left(\frac{z - Z_i(s)}{d_2}\right) \mathrm{d}N_i(s),$$

if $\widehat{\varphi}(t, z) > 0$, $\widehat{\mu}(t, z) = 0$ otherwise. It is a kernel estimator of regression function $EY(t)$, given t and z and given the fact that t is a point of corresponding point process $N(s)$. Notice, that the probability of the realization of point t depends on $Z(t^-)$. Continuity of μ at point (t, z), together with A2, suffice for the P-consistency of $\widehat{\mu}(t, z)$. The same holds for the estimate of $\mu_2(t, z) = E(Y^2(t)|Z(t) = z)$ (at point (t, z)), which is

$$\widehat{\mu}_2(t, z) = \frac{1}{\widehat{\varphi}(t, z) d_1 d_2 N(T)} \sum_{i=1}^{n} \int_0^T Y_i^2(t)\, W_1\left(\frac{t-s}{d_1}\right) W_2\left(\frac{z - Z_i(s)}{d_2}\right) \mathrm{d}N_i(s),$$

and, hence, also for $\widehat{\sigma}^2(t, z) = \widehat{\mu}_2(t, z) - (\widehat{\mu}(t, z))^2$.

3. Damage processes

In many situations, the lifetime of an object is affected by a process of growing damage. We can, for instance, imagine the process of wear, corrosion, growth of cracks, in the field of technical products reliability. Similar processes cumulating a certain important quantity influencing the risk of a crucial event can be encountered in many other fields (e. g. chemistry, environmental processes, biological and even medical studies). One of models used for the description of such a "damage process" is based on a trend function and the Wiener process describing the uncertainty, for instance

$$D(t) = g(t) + \sigma W(t).$$

The second class of models are the random sums, i. e. the point processes with random increments, cumulative processes $C(t)$. In both cases, it is assumed that there exists an upper bound B, the lifetime ends when the damage process exceeds the bound. Quite naturally, B may be a random variable, in certain instances it is not observed directly but with censoring (it will also be the case of our example). The damage modeled via $C(t)$ increases at discrete time moments, therefore these models are sometimes called the shock models. Their investigation can lead to a deeper understanding to reasons of failures, compared to a mere analysis of lifetime distribution.

In the present paper it is assumed that the most of processes are observed fully (for instance like in assumption A1), though a certain censoring is allowed. Kahle and Wendt (2000) consider also the cases when the damage processes are observed in random moments when the actual level of process is registered (the scheme of random inspections). Then, without any assumption that the number of processes observed at each small interval Δt tends to infinity, the nonparametric inference is not reliable. Nevertheless, the parametric model evaluation is still possible, with consistent results. However, Kahle and Wendth do not consider the influence of covariates, which, again, should be observed, or at least reliably predicted.

In the next example, the regression is described via the Cox's model. Its general form assumes that the hazard function can be decomposed to two factors, $h(t, z) = h_0(t) \cdot h_1(z)$, the most common form then uses $h_1(z) = \exp(\beta z)$. Cumulative baseline hazard function $H_0(t) = \int_0^t h_0(s)\,\mathrm{d}s$ is then estimated as

$$\widehat{H_0(t)} = \sum_{i=1}^{n} \int_0^t \frac{\mathrm{d}N_i(s)}{\sum_{j=1}^{n} \exp(\beta Z_j(s))\, J_j(s)},$$

while parameter β is obtained by (iterative) maximization of log partial likelihood

$$\log \mathcal{L}_p = \sum_{i=1}^{n} \int_0^T \log\Big(\frac{\exp(\beta Z_i(t))}{\sum_{j=1}^{n} \exp(\beta Z_j(t)) J_j(t)}\Big)\, \mathrm{d}N_i(t).$$

Consistency and asymptotic normality are guaranteed by the conditions of stability (the more complex variants of our A1) and by a version of Lindeberg condition – see Andersen and Gill (1982), Andersen et al (1993).

3.1. **Example.** As an artificial example with simulated data, let us consider a point process with events – failures of a car. Failures are repaired, the quantized seriousness of failure is cumulated to the cumulative process, until, finally, a non-repairable failure occurs and the lifetime of the car ends. Figure 1 a) and b) shows the processes observed for 40 cars. The reference time was the age of car, the maximal survival was about 20 years while the mean survival was 17,4052. The cars were of different year production, from new ones to cars produced 25 years ago. Figure 1 c) shows the development of the risk set, i. e. the number of cars of corresponding age remaining in the study. Several (11) trajectories of $N(t)$ and $C(t)$ end by a dot – it means that the lifetime of card ended by a non-repairable failure, while the other cars were still in use at the time of data collection. The values of $C_i(t)$ at these dots represent the distribution of upper bound B, they actually approximate the bound from below, the "right" values of the bound are censored. Nevertheless, we took the average of

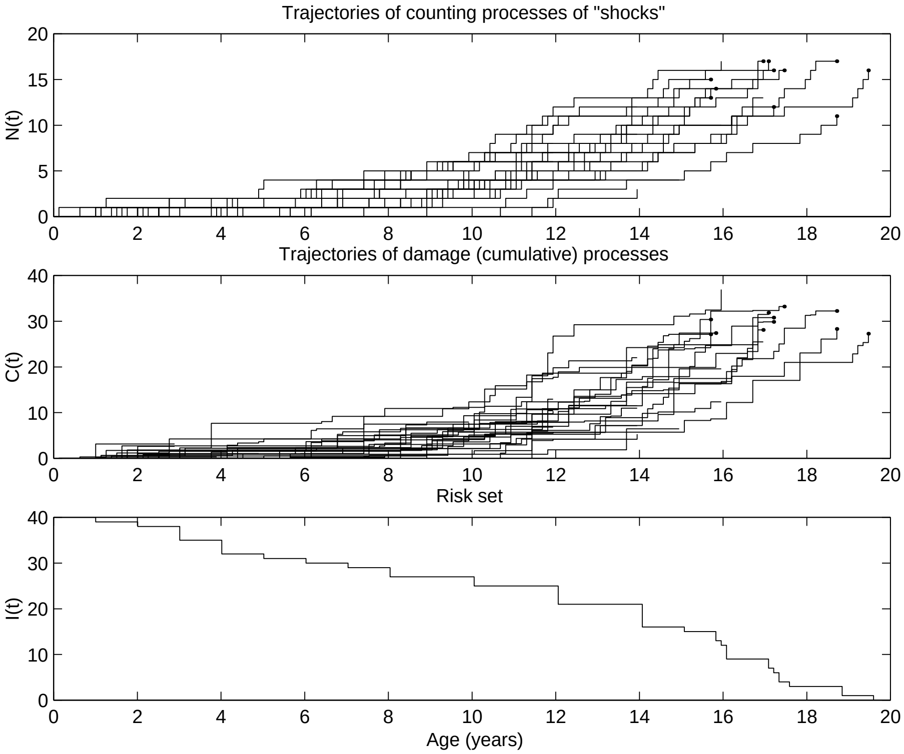

FIGURE 1. Observed processes $N_i(t), C_i(t)$ and process $J(t)$

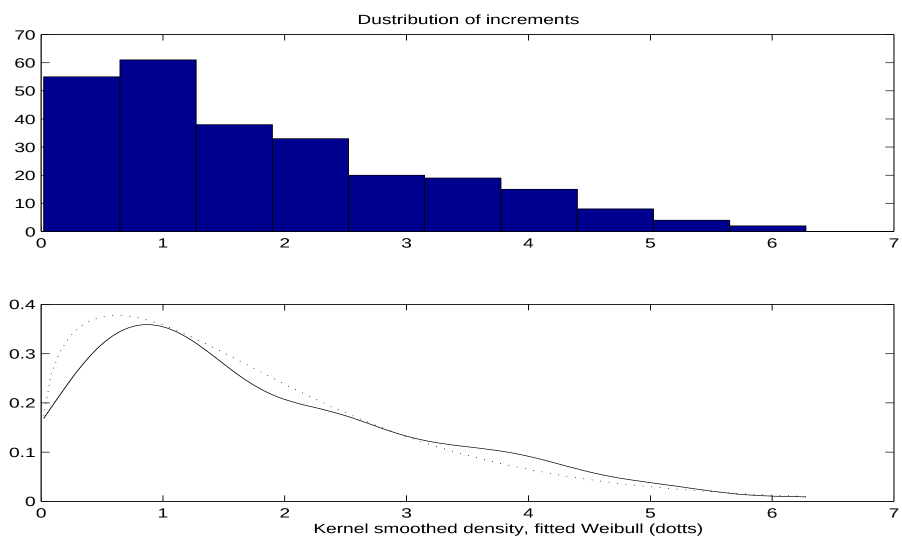

FIGURE 2. Histogram and estimated density of distribution of increments $Y(t)$

these values ($\overline{B} = 30.6130$) as a maximal damage limit used in the prediction of the fate of cars, e. g. in Figure 4.

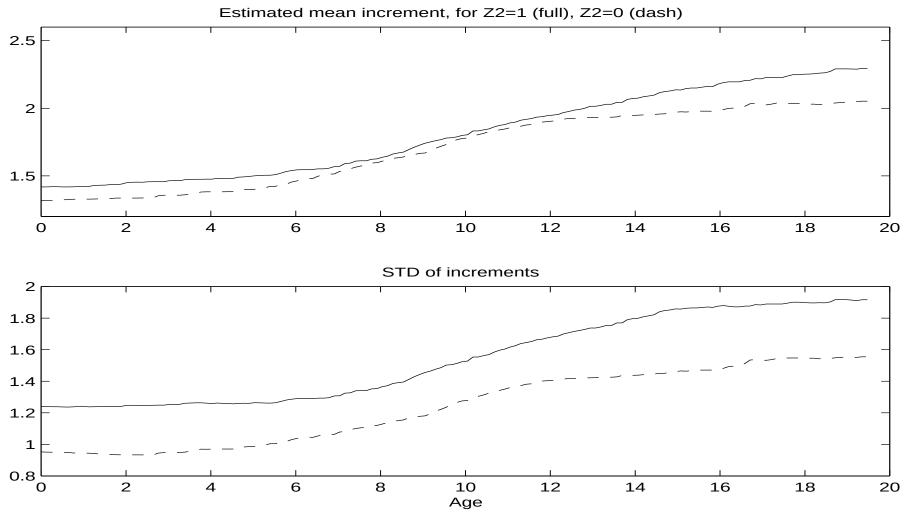

FIGURE 3. Estimated development of mean increment and STD of increments

In observed data, together 256 failures were registered. The distribution of increments is displayed in Figure 2, as a histogram and kernel-estimated density, together with the best fitted Weibull distribution. Average increment was 1.8392, estimated standard deviation 1.5039. Naturally, it was expected that both the distribution of increments $Y(t)$ and the intensity of point process $N(t)$ depended on the time and on covariates. Two covariates were considered, namely Z_1 – the year of production ($Z_1 = 1$ for 25 years old cars, $Z_1 = 25$ for new cars) and $Z_2(t)$ characterizing the conditions of car's usage, $Z_2(t) = 0$ if conditions and service was good, $Z_2(t) = 1$ if conditions were hard and/or service was bad. The dependence on these covariates was analyzed in standard Cox's regression model. Estimated parameters, with corresponding asymptotic 95 % confidence intervals, were $\beta_1 = -0.0124$ $(-0.0405,\ 0.0156)$ and $\beta_2 = 0.8306$ (0.5959, 1.0664), i.e. the second covariate was statistically significant. Figure 3 shows kernel estimates of means and standard deviations of increments, as functions of time, separately for $Z_2 = 0$ and 1 (Z_1 was not taken into account).

On the basis of these results, and for given process $Z_2(t)$, we are able to characterize (at least roughly) both components of damage process, the intensity of random point process of events and the distribution of increments – the latter for instance by Weibull distribution with means and variances depending on t and Z_2 (as in Figure 3), dependence on Z_1 being neglected. Then, we are also able to generate the trajectories of such a damage process and to predict the fate of a car. Figure 4 shows one randomly generated (predicted) trajectory of $N(t)$ and $C(t)$, for a new car and $Z_2(t) \equiv 1$, and also for 5 year old car (with actual $C(5) = 3$) and under assumption that $Z_2(t)$ will be 0 through the rest of its lifetime. Dotted line shows the upper limit of damage $\overline{B}$. Naturally, one randomly generated trajectory is not a reliable prediction. It is better to generate a large set of trajectories and to compute from them the mean trajectory, quantile intervals and prediction bands. The next section discusses such a problem.

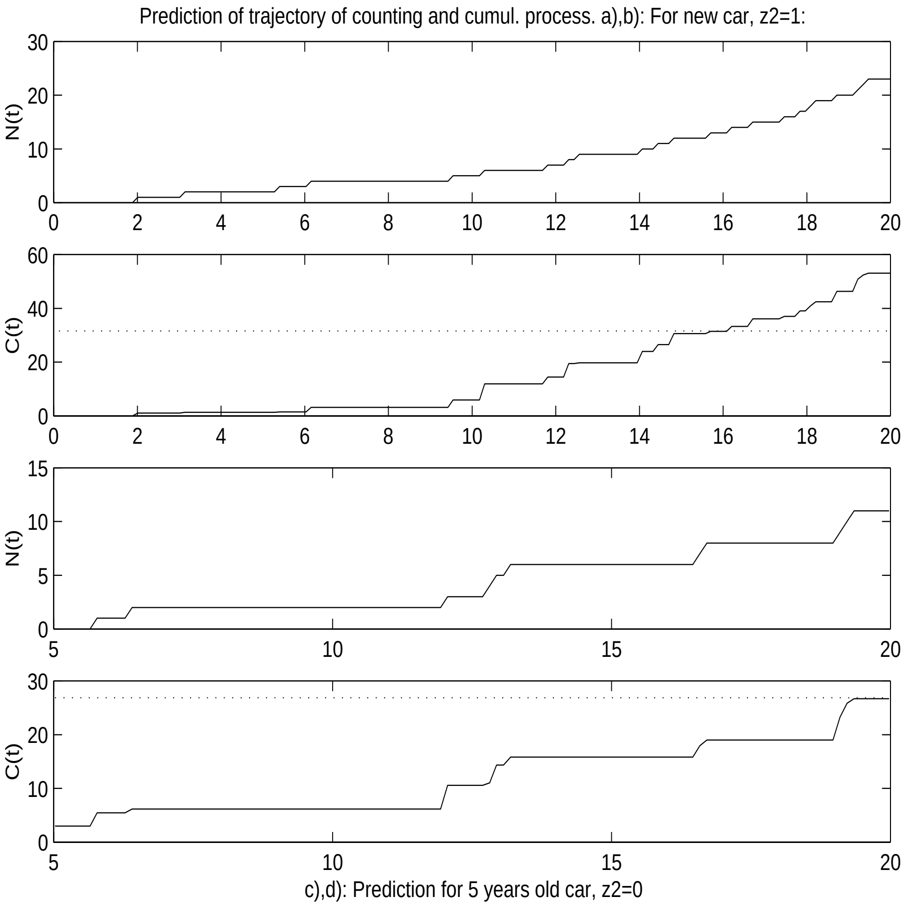

FIGURE 4. Predicted trajectories $N(t), C(t)$ for two cars

4. The problem of prediction

The asymptotic normality shown in Proposition 3 yields actually the joint confidence regions for the model functions (μ, σ, h), based on observed data, both on the whole interval $[0, T]$ and at each fixed t, too. However, we are also interested, from practical reasons, in $\alpha -$ *prediction* regions, i.e. the intervals (at fixed t) and the regions, bands for all $t \in [0, T]$, in which the trajectory of processes $N(t)$ or $C(t)$ is expected to lie with a given probability α.

For instance, insurance mathematics solves the problem of the ruin probability, which is equivalent to the problem of construction of prediction bands. In that field, mostly a simple case of compound Poisson process and the linear 'ruin band' are considered.

In the case considered here, the development of process depends on the development of covariates. So that, for reliable prediction, we need estimates of process characteristics and we also have to be able to predict relevant covariate process. Let us assume that such a prediction $z(t)$ is available, so that we can compute would-be cumulated intensity $L(t)$ (assuming that $J(t) \equiv 1$, say). Then, as regards the prediction of the counting process trajectories, the connection with Poisson process can be utilized:

α-prediction intervals for Poisson process are derived directly from α-quantiles of Poisson distribution. Intervals for counting process, conditionally for given intensity process, are obtained by a time transformation from the Poisson intervals. This transformation is possible due the continuous growth of cumulative intensity. Let T_i be times of events of a counting process with cumulated intensity $L(t)$. Then $\tau_i = L(T_i)$ are times of Poisson process with intensity 1.

Practical α-prediction band for counting process can be selected as the curve joining the end points of α^*-prediction intervals, for a conveniently chosen $\alpha^* > \alpha$. Proper α^* is obtained empirically, from analyzed or simulated data.

As regards the intervals and bands for the cumulative process, they depend on a complicated (and random) convolution of distributions of increments. We may consider an approximation depending only on cumulated means and variances of increments, though they actually depend also on the shape of distribution. A practical construction can for instance select the bound

$$b(t) = b_0(t) \cdot \{\mu(t, z(t)) + c\sigma(t, z(t))\},$$

where $b_0(t)$ is the corresponding bound (i.e. of the interval or of the band) for the counting process, c is an constant derived empirically for the actual shape of distribution of increments.

Another possibility consists in the simulation, i.e. in the "empirical" derivation of prediction regions. At a fixed point t, the empirical prediction interval is given directly by the empirical quantiles obtained from the sample of realized values $C_i(t), i = 1, 2, ..., n$. The derivation of the empirical prediction bands for trajectories of $C(t)$ on the whole interval $[0, T]$ is not so easy, though one can imagine an algorithm shifting the bands joining the empirical quantiles until, for instance, 90% of observed trajectories are inside the region.

5. Conclusion

The main purpose of the paper was to describe and analyze the random process (called here the cumulative process) consisting in the combination of the counting process with the process of random increments, and to show its application to the models of damage processes in the field of reliability analysis. Successful use of such models requires the development of the methods for estimation of the model characteristics and also the methods for the prediction of process behaviour under different conditions. Then, provided we are able to influence these conditions (i.e. covariates entering the process), we are also able to control (to slower) the growth of the damage and to prolong the lifetime of the device.

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