

ENTROPY OF FUZZY DYNAMICAL SYSTEMS

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ABSTRACT. Using fuzzy partitions instead of set partitions we have defined a new invariant of dynamical systems. In this communication we discuss the notion and show how it can be modified to be useful.

Резюме: Изучается динамическая инвариантная система которая использует "фази" разладку на месте классическои разладки множеств.

1. Fuzzy entropy

Entropy of dynamical systems has been introduced for distinguishing some non-isomorphic dynamical systems. The notion is based on the notion of the entropy

$$-\sum p_i \log p_i$$

of a set partition. Here we shall consider fuzzy dynamical systems defined by the following way.

Definition. Let $(\Omega, \mathcal{S}, P)$ be a probability space, $\mathcal{F}$ be the set of all $\mathcal{S}$-measurable functions from Ω to [0,1]. Fuzzy dynamical system is the triple $(\mathcal{F}, m, U)$, where $m : \mathcal{F} \to [0,1]$ is defined by the equality

$$m(f) = \int_\Omega f dP$$

and $U : \mathcal{F} \to \mathcal{F}$ is a maping satisfying the following conditions:

(1) If $f + g \leq 1_\Omega$, then $U(f + g) = U(f) + U(g)$.
(2) $U(1_\Omega) = 1_\Omega$.
(3) $m(U(f)) = m(f)$ for any $f \in M$.
(4) $U(f \cdot g) = U(f) \cdot U(g)$ for any $f, g \in \mathcal{F}$.

A fuzzy partition is a set $A = \{f_1, ..., f_n\} \subset \mathcal{F}$ such that

$$\sum_{i=1}^{n} f_i = 1_\Omega.$$

If $A = \{f_1, ..., f_n\}$, then $U^i(A) = \{U^i(f_1), ..., U^i(f_n)\}$, where $U^0(f) = f$, $U^{i+1}(f) = U = (U^i(f))$.

It is easy to see that $U(A)$ is a partition for any partition A. As usual define the entropy function $\varphi : [0, 1] \to [0, 1]$ by the formula

$$\varphi(x) = -x \log x,$$

if $x > 0$,

$$\varphi(0) = 0.$$

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If $A = \{f_1, ..., f_n\}, B = \{g_1, ..., g_k\}$ are two fuzzy partitions then we define

$$A \vee B = \{f_i \cdot g_j; i = 1, ..., n, j = 1, ..., k\},$$

the entropy

$$H(A) = \sum_{n=1}^{n} \varphi(m(f_i))$$

and the conditional entropy

$$H(A|B) = \sum_{i=1}^{n}\sum_{j=1}^{k} m(g_j)\varphi(\frac{m(f_i \cdot g_j)}{m(g_j)}),$$

where the summands with $m(g_j) = 0$ are ommited.

The entropy of a fuzzy dynamical system has been defined by the following formulas:

$$h(A, U) = lim_{n\to\infty}\frac{1}{n}H(\bigvee_{i=1}^{n-1} U^i(A)).$$

Finally, if $G \subset \mathcal{F}$ and $\mathcal{P}$ is the family of all fuzzy partitions, then

$$h_G(U) = sup\{h(A, U); A \in \mathcal{P}, A \subset G\}.$$

The following generalization of the celebrated Kolmogorov - Sinaj theorem holds:

Theorem. Let $C = \{C_1, ..., C_n\}$ be a measurable set partition of Ω such that the σ-algebra $\mathcal{S}$ is generated by the set $\bigcup_{i=1}^{\infty} U^i(C)$. Then for every fuzzy partition $A = \{g_1, ..., g_k\}$ there holds

$$h(A, U) \le h(C, U) + \int_{\Omega}(\sum_{i=1}^{k}\varphi(g_i))dP.$$

Of course, the definition has the following defect: If G contains all constant functions, then $h_G(U) = \infty$. This defect is eliminated by the following correction.

2. Hudetz entropy

If $A = \{f_1, ..., f_n\}$ is a fuzzy partition, then we define its Hudetz entropy

$$H^\flat(A) = \sum_{i=1}^{k}\varphi(m(f_i)) - m(\sum_{i=1}^{k}\varphi(f =_i)),$$

$$h^\flat(U) = lim_{n\to\infty}\frac{1}{n}H^\flat(\bigvee_{i=0}^{n-=1} U^i(A)),$$

and for any $G \subset \mathcal{F}$ we define

$$h^\flat_G(U) = sup\{h^\flat(A, U); A \subset G, A \in \mathcal{Q}\}$$

where $\mathcal{Q}$ is the set of all fuzzy partitions. Then the following theorem holds:

Theorem. Let $(\Omega, \mathcal{S}, P, T)$ be a dynamical system, $\mathcal{F}$ be the set of all functions $f : \Omega \to [0, 1]$ measurable with respect to $\mathcal{S}$, $U : \mathcal{F} \to \mathcal{F}$, $U(f) = f \circ T$. Let $C \subset G \subset \mathcal{F}$. Then

$$h^\flat_G(U) = h^\flat(C, U) = h(C, U).$$

3. References

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