

A CLASS OF TESTS ON THE TAIL INDEX USING THE MODIFIED EXTREME REGRESSION QUANTILES

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ABSTRAKT. Jurečková (1999) proposed a class of tests on the Pareto-type tail index of the distribution of errors in the linear regression model, based on the extreme regression quantiles. Regarding that there are not many tests on tails even in the location model, though they would be useful, we construct analogous tests on the Pareto index in the location model, modifying the *i.i.d.* observations into a two-samples model. The proposed tests are based on two extremes of a splitted sample, and on the empirical distribution functions of their mean. The main idea behind the tests is that the tails of the test criterion distinguish sharply between the heavy and light tails of the basic distribution.

The asymptotic null distribution of the test criterion is normal, the tests are consistent against the Pareto type tails with a larger index as well as against the exponential tails. The simulation study shows that the tests distinguish the tails already for moderate samples.

Abstrakt: Jurečková (2000) navrhla třídu testů indexu Paretových chvostů pro rozdělení chyb v lineárním regresním modelu. Testy je založeny na extrémních regresních kvantilech. V tomto příspěvku uvažujeme analogické testy pro model polohy. Simulační studie ukazuje, že testy rozlišují chvosty poměrně dobře i pro nepříliš rozsáhlé výběry.

Резюме: Юречкова (2000) предложила класс критериев для индекса Паретого хвоста в модели линеарной регрессии. Критерии выведёны на основе кранийх регрессыонных квантилов. В этой статье мы конструируем класс аналогичных критериев в модели сдвига. Симулированый этюд показывает что работают хорошо тоже для небольшое выборки.

1. Introduction

If we are interested in the extremal events such as the high flood levels of the rivers or extreme values of environmental indicators, then we are rather interested in the tails of the underlying distribution than in its central part. A goodness-of-fit test would not provide us with a sufficient information on the shape of tails. Our first step would consist of a decision whether the underlying distribution function F is light- or heavy-tailed; the next steps would consist of a more precise study of its shape. The model is semiparametric in its nature, involving an unknown slowly varying function, besides the real-valued parameters of interest.

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Let $Y_1, \ldots, Y_n$ be independent observations, identically distributed according to distribution function $F(y)$. We distinguish two broad classes of distributions corresponding to two types of tails of F :

- *Exponentially tailed* F (**type I**): for some $b > 0$ and $r \geq 1$, F satisfies

$$\lim_{a\to\infty} \frac{-\log(1-F(a))}{ba^r} = 1\,. \tag{1.1}$$

- *Heavy-tailed* F (**type II**): for some $m > 0$, F satisfies

$$\lim_{a\to\infty} \frac{-\log(1-F(a))}{m\log a} = 1\,. \tag{1.2}$$

A typical example of a distribution of type II is the Pareto distribution with the distribution function

$$F(x) = (1 - x^{-m})I[x \geq 1].$$

Applying the l'Hospital rule to (1.2) and taking the von Mises condition into account (see [2], Chapter 3), we get that the distributions of type II satisfy

$$1 - F(x) = x^{-m}L(x),\ x \in R \tag{1.3}$$

where $L(x)$ is a function, slowly varying at infinity.

We would like to test the hypothesis:

$$\mathbf{H}_{m_0} : F \textit{is of type II and} \ \underline{\lim}_{x\to\infty}\,[x^{m_0}(1-F(x)] \geq 1$$

against the alternative

$\mathbf{K}_{m_0}$: *the right tail of F is lighter than that of Pareto's distribution with index m_0.*

Generally, the problem of identifying the tails is semiparametric in nature, involving a nuisance slowly varying function.

Jurečková (2000) proposed a class of tests on the Pareto-type tail index of the distribution of the errors of the regression data, based on the extreme regression quantiles. Because there are not many tests in the literature, distinguishing between heavy and light tails even in the case of the i.i.d. observations, we construct analogous tests on the Pareto index in the location model, where we can apply the test proposed by Jurečková, after transforming the observations $Y_1, \ldots, Y_n$ into two samples, differing by a shift in location. A similar trick was used by Hájek (1970) for a construction of a partially adaptive rank test.

Let us modify the observations $Y_1, \ldots, Y_n,\ n = 2\nu$, in the following way: Fix $\Delta > 0$ and put

$$Y_{in}^* = Y_i + c_{in}\Delta,\ c_{in} = \begin{cases} +1, & i = 1, \ldots, \nu \\ -1, & i = \nu + 1, \ldots, n. \end{cases} \tag{1.4}$$

Hence, $Y_{1n}^*, \ldots, Y_{nn}^*$ follow the linear regression model

$$Y_{in}^* = \theta + c_{in}\Delta + U_i,\ i = 1, \ldots, n \tag{1.5}$$

with $(p = 2)$-dimensional parameter (θ, Δ), where the *i.i.d.* errors U_i have *d.f.* F (and $\theta = 0$). Following Portnoy and Jurečková (1999), we shall define the *maximal regression quantile* for the model (1.5) as the solution $(\hat\theta, \hat\Delta)$ of the minimization

$$\hat\theta = \min \qquad \textit{subject to} \qquad \hat\theta + c_{in}(\hat\Delta - \Delta) \geq Y_i,\ i = 1, \ldots, n. \tag{1.6}$$

Solution of this simple minimization takes on the form

$$\hat\theta = \frac{Y_n^{(1)} + Y_n^{(2)}}{2}, \tag{1.7}$$

$$\hat{\Delta} = \Delta + \frac{Y_n^{(1)} - Y_n^{(2)}}{2} \tag{1.8}$$

where $Y_n^{(1)} = \max\{(Y_1, \ldots, Y_\nu\}$ and $Y_n^{(2)} = \max\{Y_{\nu+1}, \ldots, Y_n\}$.

Let us repeat the sample of size n independently N times, or partition the given data set into N independent samples of sizes n. Split every sample in the above manner and calculate the extremes of the respective subsamples. The test of $\mathbf{H}_{m_0}$ is based on the empirical distribution functions of the averaged extremes of the splitted subsamples. The asymptotic power of the test is equal to 1 against the exponential tails and the test is consistent against the alternatives of lighter tails. The asymptotic null distribution of the test criterion is normal.

Another test of $\mathbf{H}_{m_0}$, based on the subsample extremes, is constructed by the same authors in [6].

The performance of the tests is illustrated on the simulated data; we see that they distinguish well the tails even for moderate samples. The proposed test is described in Section 2 along with the asymptotic null distributions of the test criterion and with a numerical illustration. The asymptotic null distribution is proved in Section 3 and Section 4 contains some consistency considerations.

2. The test and its numerical illustration

The test criterion for the hypothesis $\mathbf{H}_{m_0}$ is based on the following construction:

Observe N independent samples of the fixed sizes $n = 2\nu$ and each of them order randomly; without loss of generality, let $\mathbf{Y}_j = (Y_{j1}, \ldots, Y_{jn})'$, $j = 1, \ldots, N$ denote already the samples after random permutations. Let

$$\hat{\theta}_j = \tfrac{1}{2}(\hat{Y}_j^{(1)} + \hat{Y}_j^{(2)}), \tag{2.1}$$

$$\hat{Y}_j^{(1)} = \max(Y_{j1}, \ldots, Y_{j\nu}), \quad \hat{Y}_j^{(2)} = \max(Y_{j,\nu+1}, \ldots, Y_{jn}),\ j = 1, \ldots, N.$$

Then $(\hat{\theta}_1, \ldots, \hat{\theta}_N)$ is a random sample from distribution F^* (say); denote $\hat{F}_N^*$ the corresponding empirical distribution function, *i.e.*

$$\hat{F}_N^*(a) = \tfrac{1}{N} \sum_{j=1}^{N} I[\hat{\theta}_j \le a]. \tag{2.2}$$

Put

$$a_{N,m} = \tfrac{1}{2} N^{\frac{1-\delta}{m}} n^{\frac{1}{m}},\ 0 < \delta < 1,\ m > 0. \tag{2.3}$$

We propose the test with the critical region

$$\begin{aligned} \Big\{ (\mathbf{Y}_1, \ldots, \mathbf{Y}_N) :\ & \text{either } 1 - \hat{F}_N^*(a_{N,m_0}) = 0 \\ & \text{or } 1 - \hat{F}_N^*(a_{N,m_0}) > 0 \quad \text{and} \\ & N^{\delta/2} \Big[-\log(1 - \hat{F}_N^*(a_{N,m_0})) - (1-\delta) \log N \Big] \ge \Phi^{-1}(1-\alpha) \Big\} \end{aligned} \tag{2.4}$$

for testing $\mathbf{H}_{m_0}$ against $\mathbf{K}_{m_0}$; Φ is the standard normal *d.f.* and $\alpha \in (0,1)$ is the asymptotic significance level. The asymptotic distribution of the test criterion (2.4) under $\mathbf{H}_{m_0}$ is given in the following theorem:

Theorem 2.1. *Let* $\mathbf{Y}_j = (Y_{j1}, \ldots, Y_{jn}))'$, $j = 1, \ldots, N$ *be independent, randomly ordered samples from a population with distribution function* F *and density* f,

$F(x) < 1,\ \forall x \in \mathbb{R}$ and F being strictly monotone on the set $\{x:\ 0 < F(x) < 1\}$. Let $\hat{\theta}_1, \ldots, \hat{\theta}_N$ be the statistics defined in (2.1).

Then, if $x^{m_0}(1 - F(x)) \to 1$ as $x \to \infty$,

$$\lim_{N\to\infty} P_{m_0}\Big(0 < \hat{F}^*_N(a_{N,m_0}) < 1\Big) = 1,\ a_{N,m_0} = \tfrac{1}{2} N^{\frac{1-\delta}{m_0}} n^{\frac{1}{m_0}}, \quad 0 < \delta < 1 \tag{2.5}$$

and

$$\begin{aligned} \lim_{N\to\infty} P_{m_0}\Big\{ & N^{\delta/2}\Big[-\log(1 - \hat{F}^*_N(a_{N,m_0})) - (1-\delta)\log N\Big] \geq x, \\ & 0 < \hat{F}^*_N(a_{N,m_0}) < 1\Big\} = 1 - \Phi(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{2.6}$$

The test rejecting $\mathbf{H}_{m_0}:\ 0 < m \leq m_0$ provided

$$N^{\delta/2}[-\log(1 - \hat{F}^*_N(a_{N,m_0})) - (1-\delta)\log N] \geq \Phi^{-1}(1-\alpha) \tag{2.7}$$

has the asymptotic size α for the whole $\mathbf{H}_{m_0}$, i.e.

$$\lim_{N\to\infty} P\Big\{N^{\delta/2}[-\log(1 - \hat{F}^*_N(a_{N,m_0})) - (1-\delta)\log N] \geq \Phi^{-1}(1-\alpha)\Big\} \leq \alpha \tag{2.8}$$

for all F satisfying $\mathbf{H}_{m_0}$.

Proof: The proof of the theorem is postponed to Section 3.

Let us illustrate the performance of the test on the simulated random samples: The replications (N=25, N=50 and N=100) of samples with sizes $n = 10$, $n = 20$, $n = 50$ were simulated 1000 times from the following distributions:

Normal	$\mathcal{N}(0,1)$	$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$
Student	$t_m,\ m = 2, 5$	$f(x) = \frac{1}{\sqrt{m}\,B\left(\frac{1}{2}, \frac{m}{2}\right)} \left(1 + \frac{x^2}{m}\right)^{-(m+1)/2}, \quad x \in \mathbb{R}$
Pareto	$m = 1, 3$	$f(x) = \frac{m}{x^{m+1}}, \quad x \geq 1$
Cauchy	$(0,1)$	$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$

Notice that $\lim_{x\to\infty} x^m(1 - F(x)) = A_m = \frac{m^{\frac{m}{2}-1}}{B\left(\frac{m}{2}, \frac{1}{2}\right)}$ for the Student t_m.

Tables 1 – 4 give the numbers of non-rejections (among 1000 tests) of $\mathbf{H}_{m_0}$ for $m_0 = 1, \ldots, 5$, for the above distributions of observations, for $\delta = 0.1$ and 0.5, and for various n a N. The bold digits in the tables mean that $\mathbf{H}_{m_0}$ is satisfied for the distribution under the simulation.

Table 1.
Numbers of non-rejections of $\mathbf{H}_{m_0}$ *among 1000 cases on level* $\alpha = 0.05$
N=25, n=10

Real distribution	δ	$m_0 = 1$	$m_0 = 2$	$m_0 = 3$	$m_0 = 4$	$m_0 = 5$	$m_0 = 6$
Normal	0.1	0	0	0	0	1	2
	0.5	0	0	0	0	0	0
Student–t_2	0.1	35	428	**817**	**952**	**984**	**995**
	0.5	6	424	**819**	**938**	**970**	**984**
Student–t_5	0.1	0	3	42	155	**303**	**440**
	0.5	0	0	12	63	**136**	**220**
Pareto–m=1	0.1	**981**	**1000**	**1000**	**1000**	**1000**	**1000**
	0.5	**1000**	**1000**	**1000**	**1000**	**1000**	**1000**
Pareto–m=3	0.1	1	79	**384**	**707**	**880**	**947**
	0.5	0	58	**441**	**778**	**909**	**964**
Cauchy	0.1	694	**992**	**1000**	**1000**	**1000**	**1000**
	0.5	819	**1000**	**1000**	**1000**	**1000**	**1000**

Table 2.
Numbers of non-rejections of $\mathbf{H}_{m_0}$ *among 1000 cases on level* $\alpha = 0.05$
N=25, n=20

Real distribution	δ	$m_0 = 1$	$m_0 = 2$	$m_0 = 3$	$m_0 = 4$	$m_0 = 5$	$m_0 = 6$
Normal	0.1	0	0	0	0	2	10
	0.5	0	0	0	0	2	2
Student–t_2	0.1	60	740	**987**	**1000**	**1000**	**1000**
	0.5	76	923	**998**	**1000**	**1000**	**1000**
Student–t_5	0.1	0	5	109	403	**670**	**840**
	0.5	0	1	138	452	**712**	**847**
Pareto–m=1	0.1	**1000**	**1000**	**1000**	**1000**	**1000**	**1000**
	0.5	**1000**	**1000**	**1000**	**1000**	**1000**	**1000**
Pareto–m=3	0.1	3	191	**697**	**956**	**997**	**1000**
	0.5	1	345	**938**	**997**	**1000**	**1000**
Cauchy	0.1	932	**1000**	**1000**	**1000**	**1000**	**1000**
	0.5	997	**1000**	**1000**	**1000**	**1000**	**1000**

Table 3.
Numbers of non-rejections of $\mathbf{H}_{m_0}$ *among 1000 cases on level* $\alpha = 0.05$
N=50, n=20

Real distribution	δ	$m_0 = 1$	$m_0 = 2$	$m_0 = 3$	$m_0 = 4$	$m_0 = 5$	$m_0 = 6$
Normal	0.1	0	0	0	0	0	3
	0.5	0	0	0	0	0	0
Student–t_2	0.1	33	733	**993**	**1000**	**1000**	**1000**
	0.5	6	952	**1000**	**1000**	**1000**	**1000**
Student–t_5	0.1	0	1	74	328	**642**	**866**
	0.5	0	0	52	418	**789**	**930**
Pareto–m=1	0.1	**1000**	**1000**	**1000**	**1000**	**1000**	**1000**
	0.5	**1000**	**1000**	**1000**	**1000**	**1000**	**1000**
Pareto–m=3	0.1	0	134	**660**	**970**	**999**	**1000**
	0.5	0	168	**962**	**999**	**1000**	**1000**
Cauchy	0.1	924	**1000**	**1000**	**1000**	**1000**	**1000**
	0.5	999	**1000**	**1000**	**1000**	**1000**	**1000**

Table 4.
Numbers of non-rejections of $\mathbf{H}_{m_0}$ *among 1000 cases on level* $\alpha = 0.05$
N=100, n=50

Real distribution	δ	$m_0 = 1$	$m_0 = 2$	$m_0 = 3$	$m_0 = 4$	$m_0 = 5$	$m_0 = 6$
Normal	0.1	0	0	0	0	0	1
	0.5	0	0	0	0	0	0
Student–t_2	0.1	43	980	**1000**	**1000**	**1000**	**1000**
	0.5	24	1000	**1000**	**1000**	**1000**	**1000**
Student–t_5	0.1	0	4	129	650	**976**	**1000**
	0.5	0	0	439	999	**1000**	**1000**
Pareto–m=1	0.1	**1000**	**1000**	**1000**	**1000**	**1000**	**1000**
	0.5	**1000**	**1000**	**1000**	**1000**	**1000**	**1000**
Pareto–m=3	0.1	1	236	**959**	**1000**	**1000**	**1000**
	0.5	0	693	**1000**	**1000**	**1000**	**1000**
Cauchy	0.1	1000	**1000**	**1000**	**1000**	**1000**	**1000**
	0.5	1000	**1000**	**1000**	**1000**	**1000**	**1000**

3. Proof of the asymptotic null distribution

Let $Y_1, \ldots, Y_n$ be randomly ordered observations, $n = 2\nu$ and let $\hat{\theta}$ be defined in (1.7). Define the measure of the tail behavior of $\hat{\theta}$ as the function

$$(3.2) \qquad B^*(a) = \frac{-\log P_\theta\left(\frac{1}{2}(\hat{\theta}-\theta) \geq a\right)}{-\log(1-F(a))} = \frac{-\log P_0(Y^{(1)}+Y^{(2)} \geq 4a)}{-\log(1-F(a))}, \quad a \geq 0.$$

Similar measure was considered in [4] and later in [8] in the context of *extreme regression quantiles*, to which $\hat{\theta}$ closely relates. We shall first show that the tail behavior of $\hat{\theta}$ (1.7) distinguishes sharply between the two types of tails, while the extreme of the whole sample is insensitive to the tails.

Lemma 3.1. *Assume that the distribution function F of $Y_i - \theta,\ i = 1, \ldots, n$ satisfies $F(x) < 1,\ x \in \mathbb{R}$ and is strictly increasing on the set $\{x : 0 < F(x) < 1\}$. Then*

$$(3.3) \qquad 2^r \leq \underline{\lim}_{a\to\infty} B^*(a) \leq \overline{\lim}_{a\to\infty} B^*(a) \leq 2^{r+1} \quad \textit{for} \quad F \text{ of type } I$$
$$\lim_{a\to\infty} B^*(a) = 1 \quad \textit{for} \quad F \text{ of type } II.$$

On the other hand,

$$(3.4) \qquad \lim_{a\to\infty} \frac{-\log P_0\left(\max(Y_1, \ldots, Y_n) \geq a\right)}{-\log(1-F(a))} = 1$$

for F of both types I and II.

Proof. We have

$$(3.5) \qquad P_0(Y^{(1)}+Y^{(2)} > 4a) \leq P_0(Y^{(1)} > 2a) + P_0(Y^{(2)} > 2a) = 2(1-F^\nu(2a)),$$

hence

$$(3.6) \qquad \lim_{a\to\infty} \frac{-\log\ P_0(Y^{(1)}+Y^{(2)} > 4a)}{-\log\ (1-F(a))} \geq \lim_{a\to\infty} \frac{-\log\ (1-F(2a))}{-\log\ (1-F(a))}.$$

On the other hand,

$$(3.7) \qquad P_0(Y^{(1)}+Y^{(2)} > 4a) \geq P_0(Y^{(1)} > 2a)\ P_0(Y^{(2)} > 2a) = (1-F^\nu(2a))^2,$$

hence

$$(3.8) \qquad \lim_{a\to\infty} \frac{-\log\ P_0(Y^{(1)}+Y^{(2)} > 4a)}{-\log\ (1-F(a))} \leq \lim_{a\to\infty} \frac{-2\log\ (1-F(2a))}{-\log\ (1-F(a))}.$$

and that leads to (3.3) for F of type I and II, respectively.

For the maximum of the whole sample we get

$$(3.9) \qquad P_0\left(\max_{1\leq i\leq n} Y_i \geq a\right) = 1 - F^n(a) = (1-F(a))(1+F(a)+\ldots+F^{n-1}(a))$$

hence

$$1 - F(a) \leq P_0\left(\max_{1\leq i\leq n} Y_i \geq a\right) \leq n(1-F(a)),$$

what implies (3.4). □

Remark. Lemma 3.1 shows that the tail behavior of the sample maximum does not distinguish between types I and II.

Proof of Theorem 2.1. Let first F have exactly the Pareto tail with index m_0, i.e.

$$(3.10) \qquad \lim_{x\to\infty} [x^{m_0}(1-F(x))] = 1.$$

Then, because $\hat{\theta}_1$ is an average of two extremes, it follows e.g. from Lemma 1.3.1 in [3] (see also Lemma A3.26 in [2]) that *d.f.* F^* satisfies

$$\lim_{x\to\infty}[x^{m_0}(1-F^*(x))] = \frac{n}{2^{m_0}}. \tag{3.11}$$

$(\hat{\theta}_1,\ldots,\hat{\theta}_N)$ is a random sample from *d.f.* F^* and F^* belongs to the maximum domain of attraction of the Fréchet distribution with the distribution function

$$\Phi_{m_0}(x) = \exp\left\{-\frac{1}{x^{m_0}}\right\} I[x>0] \tag{3.12}$$

hence,

$$P\Big(\xi_N^{-1}\hat{\theta}_1 \le x\Big) \to \Phi_{m_0}(x) \tag{3.13}$$

as $N\to\infty$ with ξ_N satisfying $\lim_{N\to\infty}[N(1-F^*(\xi_N))]=1$; thus $\xi_N = \frac{1}{2}(nN)^{1/m_0}$. Denote $\hat{\theta}^{(N)} = \max\{\hat{\theta}_j,\ j=1,\ldots,N\}$ and $\hat{\theta}^{(1)} = \min\{\hat{\theta}_j,\ j=1,\ldots,N\}$. Then (3.13) implies that

$$P_m\Big(\hat{\theta}^{(N)} < a_{N,m_0}\Big) = P_m\Big(\xi_N^{-1}\hat{\theta}^{(N)} \le 2N^{-\delta/m_0}\Big) \to 0 \quad \text{as } N\to\infty \tag{3.14}$$

for a_{N,m_0} defined in (2.3). Similarly we conclude that $P_m\Big(\hat{\theta}^{(1)} > a_{N,m_0}\Big) \to 0$ as $N\to\infty$. This further implies that (2.5) holds provided F has Pareto tails with index m_0.

If $1-\hat{F}_N^*(a_{N,m_0})>0$, then

$$\begin{aligned}&N^{1/2}\left(\frac{1-F^*(a_{N,m_0})}{F^*(a_{N,m_0})}\right)^{1/2}\{-\log(1-\hat{F}_N^*(a_{N,m_0}))+\log(1-F^*(a_{N,m_0}))\}\\ &= N^{1/2}[F^*(a_{N,m_0})(1-F^*(a_{N,m_0}))]^{-1/2}[\hat{F}_N^*(a_{N,m_0})-F^*(a_{N,m_0})]\\ &\quad +\ O_p(N^{-\delta/2}).\end{aligned} \tag{3.15}$$

If we apply Theorem 4, Chapter VIII, in [7] (Cramér type large deviations) to the triangular array $(I[\hat{\theta}_1 \le 2a_{N,m_0}]-F^*(a_{N,m_0}),\ldots,I[\hat{\theta}_N \le 2a_{N,m_0}]-F^*(a_{N,m_0}))$, $N=1,2,\ldots$ and take (2.5) into account, we conclude that, given $\varepsilon>0$, there exists N_0 such that, for $N>N_0$,

$$\begin{aligned}&P_{m_0}\left\{N^{1/2}\left(\frac{1-F^*(a_{N,m_0})}{F^*(a_{N,m_0})}\right)^{1/2}\Big[-\log(1-\hat{F}_N^*(a_{N,m_0}))\right.\\ &\qquad\qquad\left.+\ \log(1-F^*(a_{N,m_0}))\Big]\ge x\right\}\\ &\le P_{m_0}\left\{N^{1/2}\left(\frac{1-F^*(a_{N,m_0})}{F^*(a_{N,m_0})}\right)^{1/2}\Big[-\log(1-\hat{F}_N^*(a_{N,m_0}))\right.\\ &\qquad\qquad\left.+\log(1-F^*(a_{N,m_0}))\Big]\ge x, 1-F^*(a_{N,m_0})>0\right\}+\varepsilon\\ &= [1-\Phi(x)](1+o(1))+\varepsilon\end{aligned} \tag{3.16}$$

for $x = o\left(N^{1/6}\right)$. Hence, regarding (3.11), we conclude

$$\begin{aligned}&P_{m_0}\left\{N^{\delta/2}[-\log(1-\hat{F}_N^*(a_{N,m_0}))-(1-\delta)\log N]\ge x,\ 1-\hat{F}_N^*(a_{N,m_0})>0\right\}\\ &\to(1-\Phi(x))\quad\text{as } N\to\infty,\ x\in\mathbb{R}\end{aligned}$$

and we arrive at (2.6). Further,

$$P_{m_0}\Big\{N^{\delta/2}[-\log(1-\hat{F}_N^*(a_{N,m_0}))-(1-\delta)\log N]\geq\Phi^{-1}(1-\alpha)\Big\}\to\alpha \tag{3.17}$$

as $N\to\infty$. If F has a heavier tail than Pareto with index m_0, then $1-F_N^*$ is ultimately stochastically larger than in the exact Pareto case, hence

$$\begin{aligned}&P\Big\{N^{\delta/2}\Big[-\log(1-\hat{F}_N^*(a_{N,m_0}))-(1-\delta)\log N\Big]\geq\Phi^{-1}(1-\alpha)\Big\}\\&\leq P_{m_0}\Big\{N^{\delta/2}\Big[-\log(1-\hat{F}_N^*(a_{N,m_0}))-(1-\delta)\log N\Big]\geq\Phi^{-1}(1-\alpha)\Big\}\to\alpha\end{aligned} \tag{3.18}$$

as $N\to\infty$; this means that the test (2.4) is of asymptotic size α for the whole hypothesis $\mathbf{H}_{m_0}$. □

4. Consistency of the test

If F has a lighter right tail than Pareto with index m_0, then $1-\hat{F}_N^*$ is ultimately stochastically smaller than in the exact Pareto case and it follows from (3.17) that

$$\begin{aligned}&\underline{\lim}_{n\to\infty}P\Big\{N^{\delta/2}\Big[-\log(1-\hat{F}_N^*(a_{N,m_0}))-(1-\delta)\log N\Big]\geq\Phi^{-1}(1-\alpha)\Big\}\\&\quad\geq\alpha,\end{aligned} \tag{4.1}$$

hence the test is asymptotically unbiased against alternative K_{m_0}.

Let now F be of type II (1.2) with index $m>m_0$; let F^* be the corresponding distribution function of $\hat{\theta}$. Then Lemma 3.1 implies that, given an $\varepsilon>0$, there exists N_0 such that, for $N>N_0$,

$$a_{N,m_0}^{-m(1+\varepsilon)}\leq 1-F^*(a_{N,m_0})\leq a_{N,m_0}^{-m(1-\varepsilon)} \tag{4.2}$$

If $1-\hat{F}_N^*(a_{N,m_0})=0$, we reject the hypothesis. Let $1-\hat{F}_N^*(a_{N,m_0})>0$; then it follows from [1] that there exists a Brownian bridge $\mathcal{B}_N$ depending on $\hat{\theta}_1,\dots,\hat{\theta}_N$ such that for $N>N_0$

$$\begin{aligned}&N^{\delta/2}\Big[-\log(1-\hat{F}_N^*(a_{N,m_0}))-(1-\delta)\log N\Big]\\&=N^{\delta/2}\Big[-\log\Big(\frac{1-\hat{F}_N^*(a_{N,m_0}))}{1-F^*(a_{N,m_0})}-1+1\Big)\Big]\\&\quad+N^{\delta/2}\Big(\frac{-\log(1-F^*(a_{N,m_0}))}{m\log a_{N,m_0}}m\log a_{N,m_0}-(1-\delta)\log N\Big)\\&\geq N^{-\frac{1-\delta}{2}}\Big|\mathcal{B}_N(F^*(a_{N,m_0}))\Big|a_{N,m_0}^{m(1-\varepsilon)}\\&\quad+(1-\delta)N^{\delta/2}\log N\left((1-\varepsilon)\frac{m}{m_0}-1\right)+O_p(N^{-\delta/2})\end{aligned} \tag{4.3}$$

hence

$$P\Big\{N^{\delta/2}\Big[-\log(1-\hat{F}_N^*(a_{N,m_0}))-(1-\delta)\log N\Big]\geq\Phi^{-1}(1-\alpha)\Big\}\to 1 \tag{4.4}$$

as $N\to\infty$, and we reject $\mathbf{H}_{m_0}$ with probability tending to 1.

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