

RCA(1) MODEL WITH HETEROSCEDASTICITY

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Abstract. The paper is concerned with a zero mean stochastic process $\{X_t\}$ that follows RCA(1) model of the form $X_t = b_t X_{t-1} + Y_t$. It is supposed that the process of random coefficients $\{b_t\}$ consists of independent random variables with constant expectation β and the second moment σ_b^2, while the error process $\{Y_t\}$ of independent random variables is permitted to be heteroscedastic. In the paper there are presented strong consistency and asymptotic normality of OLS and WLS estimators of the unknown parameter β. All these results are then generalized to the case of a non-zero mean process with unknown expectation. In the last part some simulations are presented.

Абстракт. В этой статье мы сосредоточим внимание на процессе авторегрессии первого порядка со случайным параметром. Этот протсесс имеет вид $X_t = b_t X_{t-1} + Y_t$. Сначала мы предполагаем, что X_t имеют нулевые средние значения. Процесс случайных параметров $\{b_t\}$ состоит из последовательности независимых случайных переменных со средним значением β и постоянной дисперсией σ_b^2, пока дисперсии шума Y_t зависят от t. В статье мы представляем асимптотические свойства оценок параметра β используя методы наименьших квадратов и взвешенных наименьших квадратов. Все результаты обобщены для процесса со средним значением μ и добавлены симуляциями.

1. Introduction

Random coefficient autoregressive models (RCA), which were firstly introduced by Anděl in [1], are natural generalization of AR models . Well-arranged review of results for homoscedastic RCA models is given in [10]. Through the time a lot of generalization of these basic models were studied. This paper is focused on the case when the disturbances in the RCA(1) model consist of independent random variables with heteroscedastic variances. For this model strong consistency and asymptotic normality of two types of estimates of parameter β are proved.

In a literature it can be found several papers in which asymptotic properties of estimators of unknown parameter β are studied in closely related models but none of them concerns exactly with this case. Weak consistency in heteroscedastic AR processes with independent errors is presented in [12]. Heteroscedastic AR model with martingale difference errors are studied for example in [11], [3]. While in the former one asymptotic normality is proved, in the latter one rate of convergence to normal distribution is moreover given. Further, in [9] generally explosive heteroscedastic AR processes are discussed. Fixed AR models with more general structure of heteroscedastic disturbances are studied in [5].

In the field of RCA models one can find for example [2] where authors derived rate of convergence to normal distribution in the RCA model with almost surely

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bounded and homoscedastic errors that follows martingale difference structure. In [6] asymptotic properties of estimators in the generalized RCA model in which processes of random coefficients and disturbances are permitted to be correlated are derived, but again under homoscedastic assumption. The only one paper concerned with heteroscedastic RCA models, we know about, is [8]. But strong consistency and asymptotic normality are proved under stronger assumptions there and moreover only for the OLS estimator and a zero mean case.

2. Zero mean case

2.1. **Model definition and assumptions.** Suppose that the behaviour of a stochastic process $\{X_t\}$ is described by the RCA(1) model

$$X_t = b_t X_{t-1} + Y_t, \qquad t = 1, \ldots, n. \tag{1}$$

Through the whole paper we will assume that $EX_0 = 0, 0 < EX_0^2 = \sigma_0^2 < \infty$, $Y_t, t = 1, \ldots, n$ are random variables with $EY_t = 0 \, \forall t, 0 < EY_t^2 = \sigma_t^2 < \infty$ which are independent of X_0 and that $b_t, t = 1, \ldots, n$ are random variables with $Eb_t = \beta, 0 < Eb_t^2 = \sigma_b^2 < \infty \, \forall t$ which are independent of X_0 and of $\{Y_t\}$.
It is useful to rewrite model (1) in the following way

$$X_t = \beta X_{t-1} + B_t X_{t-1} + Y_t = \beta X_{t-1} + u_t, \tag{2}$$

where $u_t = B_t X_{t-1} + Y_t$ and $B_t = b_t - \beta$. To keep unified notation let us denote $\sigma_B^2 := EB_t^2$, so the equation $\sigma_B^2 = \sigma_b^2 - \beta^2$ holds.

In this paper we will study two types of estimators of the parameter β in model (2) together with their statistical properties, namely strong consistency and asymptotic normality. We will focus on ordinary least squares (OLS) and weighted least squares (WLS) estimators which are given by (3) and (4), respectively:

$$\hat{\beta} = \frac{\sum_{t=1}^{n} X_t X_{t-1}}{\sum_{t=1}^{n} X_{t-1}^2}, \tag{3}$$

$$\hat{\beta}_W = \frac{\sum_{t=1}^{n} \frac{1}{\sigma_t^2} X_t X_{t-1}}{\sum_{t=1}^{n} \frac{1}{\sigma_t^2} X_{t-1}^2}. \tag{4}$$

Since the space for this contribution is limited, there is given only a brief summary of the most important theorems together with main ideas of their proofs. Complete proofs and all important lemmas on which the proofs are based can be found in [7].

2.2. **Strong consistency.** To establish strong consistency of the above-mentioned estimators we have to define an additional set of assumptions:

A0: both $\{Y_t\}$ and $\{b_t\}$ are processes of independent random variables,
A1: $\omega_t := E|Y_t|^{2+\delta} \leq K < \infty \, \forall t$ and for some $\delta > 0$,
A2: $\omega_b := \sup_t E|b_t|^{2+\delta} < 1$ for some $\delta > 0$,
A3: $\frac{1}{n} \sum_{t=1}^{n} \sigma_t^2 \xrightarrow[n \to \infty]{} \sigma^2 > 0$.

The proving methodology in this field is substantially based on the theory of martingale difference sequences and mixingales. Definitions of these two concepts can be found in [4]. In this case these properties are defined with respect to the filtration $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$, where $\mathcal{F}_0 = \sigma(X_0), \mathcal{F}_t = \sigma(X_0, Y_1, B_1, \ldots, Y_t, B_t)$.

Remark 2.1. *It can be shown that $\{u_t\}$ is an $\mathcal{F}_t$-martingale difference sequence ($\mathcal{F}_t$-m.d.s.). Combining this fact with (2) we could see that the random coefficient RCA(1) model (1) with independent errors can be represented as the fixed coefficient AR(1) model (2) with martingale difference errors of a structure $u_t = B_t X_{t-1} + Y_t$. But the results about these models that can be found for example in* [11] *cannot be applied directly, since the crucial assumption of this paper, i.e. $E(u_t^2|\mathcal{F}_{t-1}) = Eu_t^2$ a.s, is in our model satisfied in a degenerate case $X_t^2 = EX_t^2$ a.s, only.*

Theorem 2.1. *Under Assumptions A0–A3, $\hat{\beta} \xrightarrow[n\to\infty]{a.s.} \beta$ holds.*

Proof. Combining (2) and (3) we get $\hat{\beta} - \beta = \left(\frac{1}{n}\sum_{t=1}^{n} X_{t-1}u_t\right)\left(\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2\right)^{-1}$. In the first step it is shown that $\frac{1}{n}\sum_{t=1}^{n} X_{t-1}u_t \xrightarrow[n\to\infty]{a.s.} 0$. This arises from the fact that $\{X_{t-1}u_t\}$ is an $\mathcal{F}_t$-m.d.s. (see Lemma 1.3. in [7]) and from Theorem 20.11 in [4]. Further, it can be proved that the sequence $\{X_t^2 - EX_t^2, \mathcal{F}_t\}$ is an $L_{1+\varepsilon}$-mixingale of an arbitrary size for some $\varepsilon > 0$ (see Lemma 1.4. in [7]). This fact together with Theorem 20.16. in [4] yields that $\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 \xrightarrow[n\to\infty]{a.s.} \frac{\sigma^2}{1-(\beta^2+\sigma_B^2)} > 0$, which concludes the proof. □

In case of the WLS estimator (4) we have to add one more assumption:

A4: $0 < N \leq \sigma_t^2 \qquad \forall t.$

Theorem 2.2. *Under Assumptions A0–A4, $\hat{\beta}_W \xrightarrow[n\to\infty]{a.s.} \beta$ holds.*

Proof. The proof can be done similarly as in the previous case. To ensure existence of all a.s.-limits, it is necessary in this case to rewrite the difference $\hat{\beta}_W - \beta$ in the way $\hat{\beta}_W - \beta = \left(\frac{1}{nc_n}\sum_{t=1}^{n} \frac{1}{\sigma_t^2} X_{t-1}u_t\right)\left(\frac{1}{nc_n}\sum_{t=1}^{n} \frac{1}{\sigma_t^2}(X_{t-1}^2 - EX_{t-1}^2) + 1\right)^{-1}$, where $c_n = \frac{1}{n}\sum_{t=1}^{n} \frac{1}{\sigma_t^2} EX_{t-1}^2$. □

2.3. **Asymptotic normality.** In order to find an asymptotic distribution of given estimators of the parameter β in model (2), we have to suppose a stronger set of assumptions than in the previous paragraph. In the following let us assume:

A0: both $\{Y_t\}$ and $\{b_t\}$ are processes of independent random variables,
A1': $\eta_t := E|Y_t|^{4+\delta} \leq K < \infty \, \forall t$ and for some $\delta > 0$,
A2': $\eta_b := \sup_t E|b_t|^{4+\delta} < 1$ for some $\delta > 0$, moreover $Eb_t^4 = \gamma_b \, \forall t$,
A3: $\frac{1}{n}\sum_{t=1}^{n} \sigma_t^2 \xrightarrow[n\to\infty]{} \sigma^2 > 0$,
A5: $\frac{1}{n}\sum_{t=1}^{n} \gamma_t \xrightarrow[n\to\infty]{} \gamma$, where $\gamma_t := EY_t^4$,
A6: $\frac{1}{n}\sum_{t=1}^{n} \sigma_t^2 EX_{t-1}^2 \xrightarrow[n\to\infty]{} \overline{\sigma}^2 > 0$.

Theorem 2.3. *Under Assumptions A0, A1', A2' and A3, A5, A6, the asymptotic distribution of* $\sqrt{n}(\hat{\beta}-\beta)$ *is* $N\left(0, \Delta\left(\frac{1-(\beta^2+\sigma_B^2)}{\sigma^2}\right)^2\right)$, *where*

$$\Delta = \sigma_B^2 \frac{6(\beta^2+\sigma_B^2)\overline{\sigma}^2+\gamma}{1-\gamma_b} + \overline{\sigma}^2. \tag{5}$$

Proof. The proof is based on analyzing asymptotic behaviour of the expression $\sqrt{n}\left(\hat{\beta}-\beta\right) = \left(\frac{1}{s_n}\sum_{t=1}^{n} X_{t-1}u_t\right)\left(\sqrt{\frac{n}{s_n^2}}\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2\right)^{-1}$. Firstly, it can be derived that $\frac{1}{n}s_n^2 \xrightarrow[n\to\infty]{} \Delta$ holds for $s_n^2 := \sum_{t=1}^{n} E\left(X_{t-1}^2 u_t^2\right)$. Thus, in the rest of the proof it is sufficient to show that $\frac{1}{s_n}\sum_{t=1}^{n} X_{t-1}u_t$ has the asymptotic distribution $N(0,1)$, which requires again theory of martingale differences and mixingales (see the proof of Theorem 1.3. in [7]). □

To get asymptotic results about $\hat{\beta}_W$ we would need, next to Assumption A4, at least two additional assumptions that would guarantee the existence of the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n} E\left(\frac{1}{\sigma_t^4}X_{t-1}^2 u_t^2\right)$. The assumptions are as follows:

A7: $\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma_t^2}EX_{t-1}^2 \xrightarrow[n\to\infty]{} \underline{\sigma}^2 > 0$,

A8: $\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma_t^4}EX_{t-1}^4 \xrightarrow[n\to\infty]{} \underline{\gamma} > 0$.

Since Assumptions A6–A8 seems rather technical and complicated we formulate the following theorem with the stronger version of Assumption A3 of the form:

A3': $\sigma_n^2 \xrightarrow[n\to\infty]{} \sigma^2 > 0$.

Theorem 2.4. *Under Assumptions A0, A1', A2', A3', A4 and A5, the asymptotic distribution of* $\sqrt{n}(\hat{\beta}_W-\beta)$ *is the same as that of* $\sqrt{n}(\hat{\beta}-\beta)$ *given in Theorem 2.3.*

Proof. The proof can be done analogously as for the OLS estimator $\hat{\beta}$. Existence of all corresponding limits is ensured by Assumption A3'. □

Remark 2.2. *Under Assumption A3', all three limits defined in Assumptions A6–A8 exist and they are of the following form:*

$$\overline{\sigma}^2 = \frac{\sigma^4}{1-(\beta^2+\sigma_B^2)},$$

$$\underline{\sigma}^2 = \frac{1}{1-(\beta^2+\sigma_B^2)},$$

$$\underline{\gamma} = \frac{1}{\sigma^4}\frac{6(\beta^2+\sigma_B^2)\frac{\sigma^4}{1-(\beta^2+\sigma_B^2)}+\gamma}{1-\gamma_b}.$$

3. Non-zero mean case

A situation described by model (1) and all asymptotic results presented in Section 2 can be generalized to the case of a non-zero mean process.

3.1. **Model definition and assumptions.** Let us consider the stochastic process $\{V_t\}$ with $EV_t = \mu\ \forall t$ that follows the model

$$(V_t - \mu) = b_t\,(V_{t-1} - \mu) + Y_t, \qquad t = 1, \dots, n. \tag{6}$$

Put $X_t := V_t - \mu$ and let us suppose that all assumptions of Section 2.1 for X_t, Y_t and b_t are satisfied.

In case of known parameter μ, all previous results remain valid with $X_t := V_t - \mu$. Mostly, the parameter μ is unknown and has to be estimated. Consequently, all estimators of β have to be modified using the estimated value $\hat{\mu}$ instead of a true unknown parameter μ. In the following we will give results about strong consistency and asymptotic normality of modified OLS and WLS estimators of the form:

$$\hat{\hat{\beta}} = \frac{\sum\limits_{t=1}^{n} (V_t - \hat{\mu})(V_{t-1} - \hat{\mu})}{\sum\limits_{t=1}^{n} (V_{t-1} - \hat{\mu})^2}, \tag{7}$$

$$\hat{\hat{\beta}}_W = \frac{\sum\limits_{t=1}^{n} \frac{1}{\sigma_t^2}(V_t - \hat{\mu})(V_{t-1} - \hat{\mu})}{\sum\limits_{t=1}^{n} \frac{1}{\sigma_t^2}(V_{t-1} - \hat{\mu})^2}, \tag{8}$$

where $\hat{\mu}$ is a natural estimator of μ given by the following expression:

$$\hat{\mu} = \frac{1}{n} \sum_{t=1}^{n} V_t. \tag{9}$$

It can be proved that all asymptotic results stated in the previous section remain valid in non-zero mean case. All proofs are however a little bit complicated and they can be found in [7]. Moreover, asymptotic properties of $\hat{\mu}$ can be derived which is summarized in the sequel.

Theorem 3.1. *Under Assumptions A0–A2, $\hat{\mu} \xrightarrow[n\to\infty]{a.s.} \mu$ holds.*

Proof. Since $\hat{\mu} - \mu = \frac{1}{n} \sum\limits_{t=1}^{n} (V_t - \mu)$ the statement is a direct consequence of the fact that $\{V_t - \mu, \mathcal{F}_t\}$ is an $L_{1+\varepsilon}$-mixingale (see Lemma 1.4. in [7]) and of Theorem 20.16. in [4]. □

Remark 3.1. *In case that only strong consistency of the parameter $\hat{\mu}$ is of interest, Assumptions A1 and A2 required in Theorem 3.1 could be weakened to analogous conditions for absolute moments of an order $1+\varepsilon$ for some $\varepsilon > 0$ instead of $2+\delta$.*

Theorem 3.2. *Under Assumptions A0–A3, the asymptotic distribution of $\sqrt{n}\,(\hat{\mu} - \mu)$ is $N\left(0, \frac{\sigma^2(1+\beta)}{(1-\beta)\left(1-(\beta^2+\sigma_B^2)\right)}\right)$.*

Proof. After some algebra it can be derived that $\sqrt{n}\ (\hat{\mu} - \mu) = U_n + \left(\frac{1}{s_n} \sum\limits_{t=1}^{n} \rho_{n,t} u_t\right) \left(\sqrt{\frac{n}{s_n^2}}\right)^{-1}$, where $\rho_{n,t} := \frac{1-\beta^{n-t+1}}{1-\beta}$, $s_n^2 := \sum\limits_{t=1}^{n} \rho_{n,t}^2 E u_t^2$ and

U_n is a random variable for which $U_n \xrightarrow[n\to\infty]{p} 0$ holds. Firstly, it is shown that $\frac{1}{n}s_n^2 \xrightarrow[n\to\infty]{} \frac{\sigma^2}{1-\beta}\left[\frac{1+\beta}{1-(\beta^2+\sigma_B^2)}\right]$. Further, it has to be proved that $\frac{1}{s_n}\sum_{t=1}^{n}\rho_{n,t}u_t$ has the limiting distribution $N(0,1)$ (see the proof of Theorem 2.4. in [7] for more details). □

4. Simulations

To demonstrate asymptotic behaviour of previously studied estimators we present here results of a short simulation study.

The error process $\{Y_t\}$ considered in (1) was generated from the distribution $N(0,\sigma_t^2)$ with σ_t^2 chosen to satisfy one of the following models (sample size in each case was 500 observations):

H1: homoscedastic with $\sigma_t^2 = 3$ for $t = 1,\ldots,500$,
H2: heteroscedastic with stepwise ascending σ_t^2 having successively values 1, 1.5, 2.5, 4 with breaking points at times $n = 80, 180, 300$,
H3: heteroscedastic with linearly descending σ_t^2 given by $\sigma_t^2 = 6 - 0.01t$ for $t = 1,\ldots,400$ and then constant at the value 2.

For the process of random parameters $\{b_t\}$ normal respectively uniform distribution were supposed, exact parameters of which are given in Table 1. These parameters were chosen in order to study the influence of the stability condition $\beta^2 + \sigma_B^2 < 1$ on one side, and the individual impact of β^2 respectively of σ_B^2 for the fixed value of $\beta^2 + \sigma_B^2$ on the other side.

case	A	B	C	D	E	F
b_t	N(0.2;0.2)	N(0.5;0.7)	N(0.75;0.2)	N(0.25;0.7)	U(-1;1)	U(0;1)
β	0.2	0.5	0.75	0.25	0	0.5
σ_B^2	0.2	0.7	0.2	0.7	1/3	1/12
$\beta^2+\sigma_B^2$	0.24	0.95	0.7625	0.7625	1/3	1/3

Table 1: Parameters of distribution of b_t

Combining all these possibilities we simulated 18 different types of model (1). Convergence of estimates towards the true parameter β was studied for generated series with ascending number of observations $n = 50, 100, 150, \ldots, 500$. In each case estimated parameters were based on 50 independent realizations.

In addition to the previous part of the paper, in this section we also consider conditionally weighted least squares estimator (CWLS), which is defined as

$$\hat\beta_{CW} = \sum_{t=1}^{n}\frac{X_tX_{t-1}}{\sigma_B^2X_{t-1}^2+\sigma_t^2}\Bigg/\sum_{t=1}^{n}\frac{X_{t-1}^2}{\sigma_B^2X_{t-1}^2+\sigma_t^2}. \tag{10}$$

Since $E(u_t^2|\mathcal{F}_{t-1}) = \sigma_B^2X_{t-1}^2+\sigma_t^2$, weights in (10) should better correspond to character of heteroscedasticity in model (2) rather than σ_t^2 alone. This hypothesis is numerically demonstrated by our results, while the theoretical derivation of asymptotic properties of $\hat{\beta}_{CW}$ is a subject of the future research. For generalized RCA(p) model with homoscedastic disturbances it has been already proved in [6].

Graphs 4–4 demonstrate convergence of all estimates of β together with development of their estimated variances in cases C and D and under all types of heteroscedasticity H1, H2 and H3. Summary of selected results for $n = 50, 200, 500$ in remaining cases is given in Table 2.

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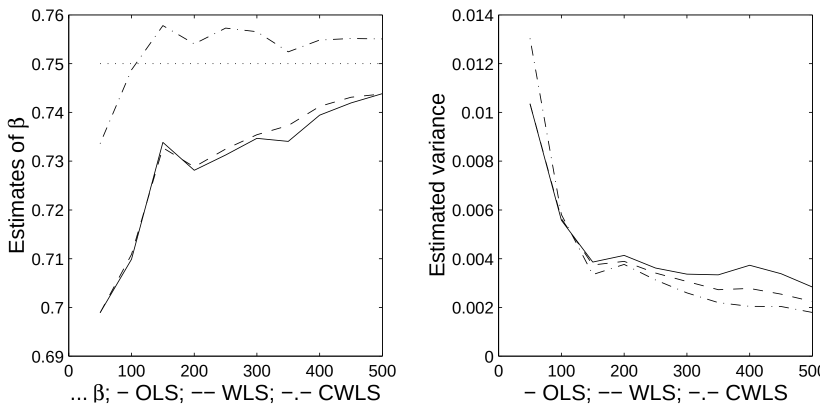
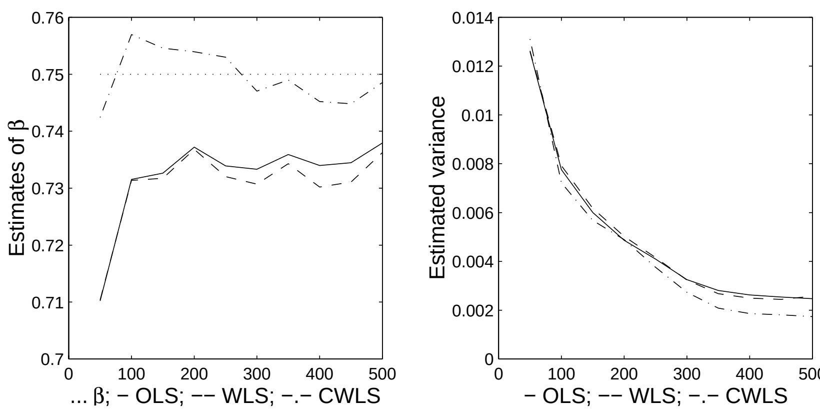
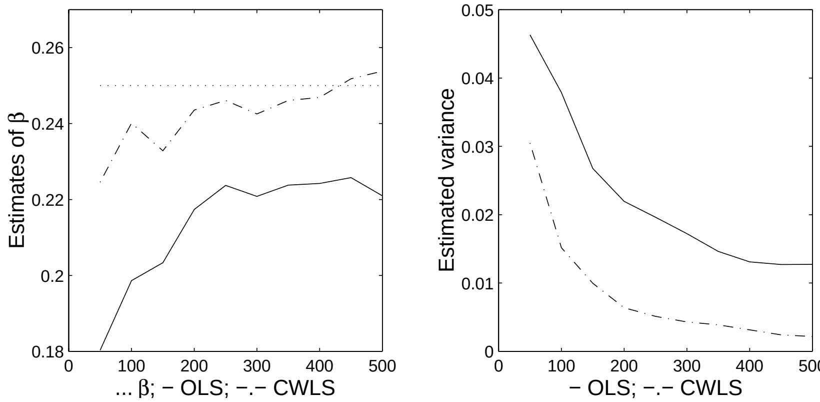
Graph 0: $b_t \sim N(0.75; 0.2)$, homoscedasticity H1



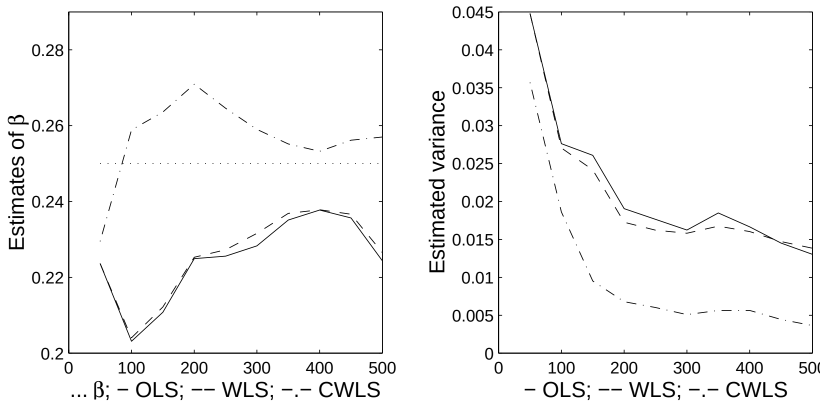
Graph 0: $b_t \sim N(0.75; 0.2)$, heteroscedasticity H2



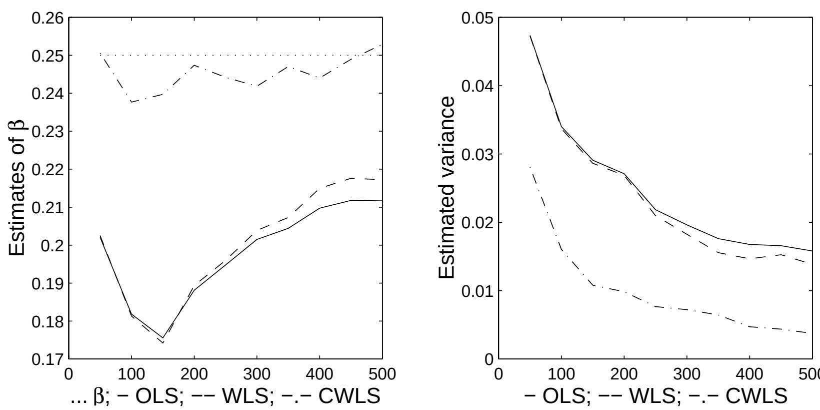
Graph 0: $b_t \sim N(0.75; 0.2)$, heteroscedasticity H3



Graph 0: $b_t \sim N(0.25; 0.7)$, homoscedasticity H1



Graph 0: $b_t \sim N(0.25; 0.7)$, heteroscedasticity H2



Graph 0: $b_t \sim N(0.25; 0.7)$, heteroscedasticity H3

From numerical and graphical results it is clearly seen that in all cases WLS, estimates are slightly preferable to OLS ones in both criteria convergence and estimated variance. Improvement of CWLS estimates is much more significant than that of WLS ones.

In case A, the differences between weighted and unweighted estimates and their estimated variances are not so big since both β and σ_B^2 are relatively small. On the other hand in case B, where both parameters are quite big, better convergence and smaller estimated variances of CWLS estimates are evident. It is well known that if the value of stability condition is close to 1, both OLS and WLS estimates can be unstable. In this light very good behaviour of CWLS estimates in case B could seem quite surprising.

Results of cases C and D demonstrate that under the fixed value of $\beta^2+\sigma_B^2$, higher values of σ_B^2 lead to greater improvement of estimated variances of WLS and mainly CWLS estimates in comparison with OLS ones. Benefit from estimated variances of weighted estimates is not as big as in case C where σ_B^2 is small.

In cases of uniform distribution, the situation is analogous with the only one difference, namely that better convergence of CWLS estimates is not as significant as in cases of normal distribution.

			N(0.2;0.2)		N(0.5;0.7)		U(-1;1)		U(0;1)	
		n	$\hat{\beta}_*$	$\hat{s}^2_{\hat{\beta}_*}$	$\hat{\beta}_*$	$\hat{s}^2_{\hat{\beta}_*}$	$\hat{\beta}_*$	$\hat{s}^2_{\hat{\beta}_*}$	$\hat{\beta}_*$	$\hat{s}^2_{\hat{\beta}_*}$
H1	OLS	50	0.1565	0.0219	0.3684	0.0407	-0.0315	0.0263	0.4511	0.0226
		200	0.1736	0.0068	0.4413	0.0221	0.0055	0.0093	0.4844	0.0061
		500	0.1880	0.0022	0.4527	0.0161	0.0032	0.0038	0.4955	0.0020
	CWLS	50	0.1667	0.0212	0.4707	0.0238	-0.0387	0.0184	0.4558	0.0233
		200	0.1782	0.0070	0.4892	0.0068	-0.0018	0.0064	0.4865	0.0062
		500	0.1926	0.0020	0.5058	0.0030	-0.0002	0.0024	0.4960	0.0018
H2	OLS	50	0.1895	0.0191	0.3753	0.0442	-0.0117	0.0241	0.4737	0.0143
		200	0.1941	0.0059	0.4269	0.0260	-0.0004	0.0073	0.5084	0.0055
		500	0.1935	0.0035	0.4525	0.0150	0.0007	0.0036	0.4986	0.0026
	WLS	50	0.1895	0.0191	0.3753	0.0442	-0.0117	0.0241	0.4737	0.0143
		200	0.1928	0.0044	0.4306	0.0238	0.0031	0.0069	0.5068	0.0048
		500	0.1948	0.0024	0.4536	0.0157	0.0031	0.0028	0.5011	0.0021
	CWLS	50	0.1946	0.0188	0.5047	0.0311	-0.0082	0.0218	0.4824	0.0131
		200	0.2004	0.0042	0.5033	0.0085	0.0161	0.0055	0.5117	0.0047
		500	0.2032	0.0021	0.5048	0.0034	0.0087	0.0030	0.5049	0.0022
H3	OLS	50	0.2205	0.0176	0.4108	0.0345	0.0408	0.0238	0.5260	0.0160
		200	0.2023	0.0050	0.4457	0.0228	0.0132	0.0100	0.5075	0.0033
		500	0.1958	0.0025	0.4397	0.0145	0.0068	0.0036	0.5011	0.0013
	WLS	50	0.2206	0.0178	0.4105	0.0346	0.0411	0.0237	0.5264	0.0159
		200	0.2012	0.0050	0.4423	0.0245	0.0113	0.0102	0.5055	0.0034
		500	0.1975	0.0025	0.4388	0.0128	0.0059	0.0031	0.4994	0.0013
	CWLS	50	0.2271	0.0173	0.5503	0.0277	0.0288	0.0237	0.5331	0.0169
		200	0.1968	0.0047	0.5212	0.0078	0.0143	0.0072	0.5069	0.0035
		500	0.2001	0.0027	0.5091	0.0027	0.0038	0.0028	0.4997	0.0014

Table 2: Estimates of parameters β and their estimated variances

In practice we rarely know the true values of variances σ_t^2 and σ_B^2 and hence we could not compute estimators $\hat{\beta}_W$ and $\hat{\beta}_{CW}$ directly. However, we can use the two-stage estimation procedure. It the first stage, OLS estimate $\hat{\beta}$ is computed and then unknown variances σ_t^2 and σ_B^2 are taken as OLS estimates of parameters S_t and S_B from the regression model of the form $\hat{u}_t^2 = S_t + S_B X_{t-1}^2 + \xi_t$,where $\hat{u}_t$ denotes OLS residuals from the first stage. In the second stage, WLS and CWLS estimates are then computed using values $\hat{\sigma}_t^2$ and $\hat{\sigma}_B^2$ instead of unknown σ_t^2 and σ_B^2.

We also applied this two-stage procedure to all 18 types of models in order to compare estimated variances with the true ones and subsequently to compare WLS and CWLS estimates of β computed using true and estimated variances.

From our simulation results (which are not presented here) it can be seen that the biggest differences between $\hat{\sigma}_t^2$, σ_t^2 and $\hat{\sigma}_B^2$, σ_B^2 are realized in cases B and D, where the random coefficients have quite high variance. In the remaining cases the estimates are very accurate. We can also see that estimates $\hat{\sigma}_t^2$ always overestimate true values σ_t^2 in contrast to $\hat{\sigma}_B^2$ that are lower than σ_B^2 in all cases. It can lead to the conclusion that this procedure is not able to separate well the variation in the data due to the variation in coefficients and due to random errors. It seems that it moves certain part of the variation σ_B^2 into the variation σ_t^2. On the other hand, an interesting observation could be the fact that differences in estimated variances from the first stage do not have as significant impact on the second-stage WLS and CWLS estimates of β as we have expected, not even in cases B and D. This is in our opinion a positive result, since it allows us to compute weighted estimates based on estimated variances without committing any significant deviations from truly computed WLS and CWLS ones.

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