

A NOTE ON CONVERGENCE OF LOCAL MARTINGALES

PETR DOSTÁL

ABSTRACT. It is investigated the topology of $\mathcal{L}(\mathbb{L})$, the set of all distributions of pairs (X, Y) such that Y is a real continuous local martingale on the canonical filtration of the process X and X is stochastic process on some separable metric space T.

Абстракт. В этой статьи мы занимаемся топологией множества всех распределений пар (X, Y), где Y обозначает реальный непрерывный локальный $\mathcal{F}_t^X$-мартингал и X стохастический процесс со значениами в сепарабельном метрическом пространстве T.

1. Introduction

We continue the research on topology of $\mathcal{L}(\mathbb{L})$ that was started in [3] and generalize the following result from [3].

(1) $$\mathcal{L}(\mathbb{L}) \text{ is a relatively weakly closed set in } \mathcal{L}(\mathbb{A})$$

where $\mathcal{L}(\mathbb{A})$ denotes the set of all probability distributions of pairs (X, Y) such that Y is a real-valued continuous X-adapted process if we consider only continuous processes X. We offer another idea how to prove and generalize the above result. (see Theorem 1, Theorem 5)

2. Motivation

We refer to [3] for inspiration but we only introduce one example which is denoted in [3] as 1.2 Example.

I. The set $\mathcal{W}$ of all probability distributions of processes X such that there exists a weak solution $(\Omega, \mathcal{F}, P, \mathcal{F}_t, W, X)$ of a stochastic differential equation (SDE)

(2) $$dX_t = b(t, X)dt + \sigma(t, X)dW_t$$

is a Borel convex set in the set of all Borel probability measures on $\mathbb{C}(\mathbb{R}^+)$ for arbitrary progressive coefficients (progressive coefficients will be defined in section 3 b and σ and weakly closed set in $\mathcal{P}(B)$, the set of all Borel probability measures on $\mathbb{C}(\mathbb{R}^+)$ concentrated on

(3) $$B = \{x \in \mathbb{C}(\mathbb{R}^+), \int_0^t |b(s, x_s)| + \sigma^2(s, x_s)ds < \infty \ \forall t \geq 0\} \in \mathcal{B}(\mathbb{C}(\mathbb{R}^+)).$$

Especially, $\mathcal{W}$ is a weakly closed convex set in $\mathcal{P}(B)$ if $b(t, .), \sigma(t, .) : \mathbb{C}(\mathbb{R}^+) \to \mathbb{R}$ are continuous mappings for every $t \geq 0$ and bounded on compact sets in $\mathbb{R}^+ \times \mathbb{C}(\mathbb{R}^+)$.

2000 *Mathematics Subject Classification.* Primary 62B05; Secondary 28C20.

Key words and phrases. Local martingales, convergence, canonical filtration.

The paper was prepared with the support of the Research program MSM 113200008.

Note only or see 1.2 Example in [3] that if there is a weak solution $(\Omega, \mathcal{F}, P, \mathcal{F}_t, W, X)$ of a SDE (2) then

$$X \in B \text{ a.s. and } G_b(X,t), G_{b,\sigma}(X,t) \text{ are local martingales} \tag{4}$$

with respect to completed canonical filtration of the process X where

$$G_b(x,t) = [x_t - x_0 - \int_0^t b(s,x_s)ds] \cdot I_B \text{ and} \tag{5}$$

$$G_{b,\sigma}(x,t) = [x_t^2 - x_0^2 - \int_0^t 2x_s b(s,x_s) - \sigma^2(s,x_s)ds] \cdot I_B. \tag{6}$$

Conversely, if (4) holds, then there exists a weak solution $(\Omega^*, \mathcal{F}^*, P^*, \mathcal{F}_t^*, W^*, X^*)$ of a SDE (2) such that X and X^* have the same distribution.

II. If $G_b, G_{b,\sigma}$ are continuous and SDE (1) is such that there exists its weak solution for any deterministic initial condition, then the equation possesses a weak solution with arbitrary initial condition in $\mathcal{P}(\mathbb{R})$, the set of all Borel probability measures on $\mathbb{R}$.

3. Notation

For $f : \mathbb{R}^+ \to \mathbb{R}$ continuous function started at $0, c > 0$ we denote by

$$\alpha_c(f) := f(. \wedge \tau_c^f), \text{ where } \tau_c^f = \inf\{t \geq 0, |f| \geq c\} \tag{7}$$

is the time of the first entry of the function $|f|$ into the set $\{-c, c\}$. This transformation gives us a Borel measurable mapping $\alpha_c^0 : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+), \alpha_c^0(g) = \alpha_c(g - g(0))$ where $\mathbb{C}(\mathbb{R}^+)$ denotes the set of all continuous functions on $\mathbb{R}^+$ endowed with the topology of uniform convergence on compact intervals. Recall that a real-valued continuous process Y is an $\mathcal{F}_t^X$-local martingale iff for every $c > 0$ $\alpha_c^0(Y)$ is an $\mathcal{F}_t^X$-martingale i.e. Y is $\mathcal{F}_t^X$-adapted process and for $s \leq t$

$$E^{\mathcal{F}_s^X} \alpha_c^0(Y)_t = \alpha_c^0(Y)_s \text{ a.s.} \tag{8}$$

where $\mathcal{F}_t^X$ denotes the canonical filtration of the process X defined as follows

$$\mathcal{F}_t^X = \sigma\{[X_u \in B], u \leq t, B \in \mathcal{B}(\mathbb{R})\} \tag{9}$$

where $\sigma(\mathcal{A})$ denotes the smallest σ-algebra which contains $\mathcal{A}$ as a subset. Recall that $\mathcal{B}(S)$ denotes the Borel σ-algebra on metric space S. We promised in section 2 to define progressive coefficients. Say that $b : \mathbb{C}(\mathbb{R}^+) \times \mathbb{R}^+ \to \mathbb{R}$ is a progressive coefficient if there are some $b_t : \mathbb{C}[0,t] \times [0,t] \to \mathbb{R}, t \geq 0$ Borel measurable functions such that

$$\text{for } x \in \mathbb{C}(\mathbb{R}^+), u \leq t \quad b_t(x_{s,s\leq t}, u) = b(x,u) \tag{10}$$

where $\mathbb{C}[0,t]$ denotes the set of all continuous functions on $[0,t]$ endowed with the topology of uniform convergence.

4. Theorems

Recall that X is a real-valued continuous process iff it is a random variable with values in $(\mathbb{C}(\mathbb{R}^+), \mathcal{B}(\mathbb{C}(\mathbb{R}^+)))$.

Theorem 1. *Let Y_n, Y be real-valued continuous processes such that $Y_n \to Y$ a.s. in $\mathbb{C}(\mathbb{R}^+)$. Then for $c > 0$ there are sequences $\delta_k \in (0, c)$ and $n_k \in \mathbb{N}$ such that*

$$\alpha^0_{c-\delta_k}(Y_{n_k}) \to \alpha^0_c(Y), k \to \infty \text{ a.s. in } \mathbb{C}(\mathbb{R}^+). \tag{11}$$

Proof: see thm. 1 in [2]. A combination of this theorem and Skorochod Theorem yields Theorem 2.

Skorochod Theorem. *Suppose that P_n, P are Borel probability measures on metric space S. If $P_n \to P$ weakly as $n \to \infty$ and P has a separable support, then there exist random elements X_n and X, defined on a common probability space $(\Omega, \mathcal{F}, P)$, such that*

$$\mathcal{L}(X_n) = P_n, \mathcal{L}(X) = P \text{ and } X_n(\omega) \to X(\omega) \text{ for every } \omega \in \Omega \text{ as } n \to \infty \tag{12}$$

where $\mathcal{L}(Z)$ denotes the probability distribution of a random variable Z. (see thm. 6.7. page 70 in [1])

Theorem 2. *Let X, X_n be real-valued continuous processes such that $X_n \to X$ in distribution as $n \to \infty$ in $\mathbb{C}(\mathbb{R}^+)$. If $R : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+)$ is a continuous mapping such that for $n \in \mathbb{N}$ $R(X_n)$ is a local $\mathcal{F}^{X_n}_t$ martingale and $R(X)$ is an $\mathcal{F}^X_t$-adapted process, then $R(Y)$ is also an $\mathcal{F}^X_t$-local martingale.*

Proof: See thm. 2 in [2].

The method given by Theorem 1. seems to be sufficient for generalization but a little bit unpleasant. One may ask if there is some better approximation procedure of $\alpha_c(Y)$ by $\alpha_d(Y_k)$ for $Y_k \to Y$ in $\mathbb{C}(\mathbb{R}^+)$. The answer is given by

Theorem 3. *Let Y_n, Y be real-valued continuous processes such that $Y_n \to Y$ a.s. as $n \to \infty$ in $\mathbb{C}(\mathbb{R}^+)$ and for $n \in \mathbb{N}$ Y_n is a local $\mathcal{F}^{Y_n}_t$-martingale. Then for $c > 0$ holds*

$$\alpha^0_c(Y_n) \to \alpha^0_c(Y) \text{ a.s. as } n \to \infty \text{ in } \mathbb{C}(\mathbb{R}^+). \tag{13}$$

Remark. Recall only that Y is an $\mathcal{F}^Y_t$ local martingale iff Y is an $\mathcal{F}_t$-martingale for some filtration $(\mathcal{F}_t, t \geq 0)$.

Proof of Theorem 3 is based on Theorem 2 and 4 which says that it is enough to show that Y is a local $\mathcal{F}^Y_t$-martingale. Now put $R :=$ id and use Theorem 2.

Theorem 4. *If Y is an $\mathcal{F}^Y_t$-local martingale, $c > 0$, then the mapping α^0_c is continuous at $Y(\omega)$ for a.s. $\omega \in \Omega$.*

Proof: See thm. 3 in [2]

Our generalization of result (1) reads as follows.

Theorem 5. *Let X_n, X be T-valued stochastic processes for some separable metric space T; Y_n, Y be $\mathbb{R}$-valued continuous processes such that for $n \in \mathbb{N}$ Y_n is an $\mathcal{F}_t^{X_n}$-local martingale and Y is an $\mathcal{F}_t^X$-adapted process. If moreover finite dimensional distributions of X_n associated with Y_n converge to finite dimensional distribution of X associated with Y i.e. for f continuous bounded function on $T^k \times \mathbb{C}(\mathbb{R}^+)$, $u \in (R^+)^k, k \in \mathbb{N}$*

$$Ef(X_n(u_1), \cdots, X_n(u_k), Y_n) \to Ef(X(u_1), \cdots, X(u_k), Y) \text{ as } n \to \infty. \tag{14}$$

Then Y is an $\mathcal{F}_t^X$-local martingale.

Proof: see Proposition in [2] which is a little bit more general but its formulation is more unpleasant.

One may ask if we could leave out the assumption, Y is an $\mathcal{F}_t^X$-adapted process. The following example shows that it is impossible even in the situation of Theorem 2 where $Y_n = R(X_n)$ and $Y = R(X)$.

5. Example

We will find $\mathbb{R}$-valued continuous processes such that $X_n \to X$ as $n \to \infty$ everywhere and $R : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+)$ a continuous mapping such that for $n \in \mathbb{N}$ $R(X_n)$ is an $\mathcal{F}_t^{X_n}$-local martingale but $R(X)$ is not even adapted to the completed canonical filtration of the process X.

Denote by W one-dimensional Brownian motion and for $n \in \mathbb{N}$ put

$$X_n(t) = W(t \wedge 1)/n + W[(t-1)^+ \wedge 1] \to X(t) = W[(t-1)^+ \wedge 1], n \to \infty$$

$$\text{and } R : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+),\ R(f)(t) = f(t+1) - f(1).$$

Now it is enough to check that $R(X_n) = W(. \wedge 1)$ is an $\mathcal{F}_t^{X_n} = \mathcal{F}_{t\wedge 1}^W$-local martingale but $R(X) = W(. \wedge 1)$ at time 1 is a nontrivial random variable and it cannot be measurable with respect to the completion of a trivial σ-algebra $\mathcal{F}_t^X = \mathcal{F}_{(t-1)^+\wedge 1}^W$ at time $t = 1$.

6. References

[1] P. Billingsley (1999): Convergence of Probability Measures, J.Wiley, New York.

[2] P. Dostál (2001): A Note on Weak Convergence of Martingale Measures (to be published)

[3] J. Štěpán, P. Ševčík (2000): Local Martingale Measures. Acta Universitatis Carolinae, Mathematica et Physica, vol. 41, No 1, Praha.

Charles Univ. of Prague, Dept. pf Probability and Statistics, Sokolovská 83, CZ – 186 75 Praha 8 – Karlín, Czech Republic

E-mail address: dostal@karlin.mff.cuni.cz; petdos@post.cz