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SAMPLE PATH BEHAVIOUR OF WIENER AND CAUCHY PROCESSES

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Abstrakt. Článek popisuje chování trajektorií Wienerova a Cauchyho procesu pomocí integrálních testů.
Резюме: В стате описаны интеграљные тэсты которые характеризуют траектории процессов Броуна и Коши.

1. Definitions and notations

Let $\{V(t);\ t \geq 0\}$ be a stochastic process. We define the following classes of functions (cf. Révész (1990), pp. 33–34).

Definition 1.1. The function ψ belongs to the *upper-upper class* of $\{V(t);$ $t \geq 0\}$ $(\psi(t) \in \mathrm{UUC}(V(t)))$ if for almost all $\omega \in \Omega$ there exists a $t_0 = t_0(\omega) > 0$ such that $V(t) < \psi(t)$ for all $t > t_0$.

Definition 1.2. The function ψ belongs to the *upper-lower class* of $\{V(t);$ $t \geq 0\}$ $(\psi(t) \in \mathrm{ULC}(V(t)))$ if for almost all $\omega \in \Omega$ there exists a sequence of positive numbers $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \cdots$ with $t_n \to \infty$ such that $V(t_i) \geq \psi(t_i),\ i = 1, 2, \ldots$.

Definition 1.3. The function ψ belongs to the *lower-upper class* of $\{V(t);$ $t \geq 0\}$ $(\psi(t) \in \mathrm{LUC}(V(t)))$ if for almost all $\omega \in \Omega$ there exists a sequence of positive numbers $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \cdots$ with $t_n \to \infty$ such that $V(t_i) \leq \psi(t_i),\ i = 1, 2, \ldots$.

Definition 1.4. The function ψ belongs to the *lower-lower class* of $\{V(t);$ $t \geq 0\}$ $(\psi(t) \in \mathrm{LLC}(V(t)))$ if for almost all $\omega \in \Omega$ there exists a $t_0 = t_0(\omega) > 0$ such that $V(t) > \psi(t)$ for all $t > t_0$.

Assume, for a moment, that a process $\{V(t);\ t \geq 0\}$ and functions $\psi(t)$ and $\xi(t)$ are positive. If

$$\limsup_{t\to\infty} \frac{V(t)}{\psi(t)} = 1 \quad \text{a.s.},$$

then for every $\varepsilon > 0$, $(1+\varepsilon)\psi(t) \in \mathrm{UUC}\big(V(t)\big)$ and $(1-\varepsilon)\psi(t) \in \mathrm{ULC}\big(V(t)\big)$. Similarly, if

$$\liminf_{t\to\infty} \frac{V(t)}{\xi(t)} = 1 \quad \text{a.s.},$$

then for every $\varepsilon > 0$, $(1+\varepsilon)\xi(t) \in \mathrm{LUC}\big(V(t)\big)$ and $(1-\varepsilon)\xi(t) \in \mathrm{LLC}\big(V(t)\big)$. But we do not know whether $\psi(t)$ belongs to $\mathrm{UUC}\big(V(t)\big)$ or to $\mathrm{ULC}\big(V(t)\big)$

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and whether $\xi(t)$ belongs to $\mathrm{LLC}\big(V(t)\big)$ or to $\mathrm{LUC}\big(V(t)\big)$. Therefore, the division of functions into UUC and ULC classes gives a stronger result than a lim sup result, and the division of functions into LLC and LUC classes gives a stronger result than a lim inf result.

Define the following class of functions,

$$\begin{aligned} \Psi = \{\psi;\ &\psi \text{ is a positive function defined in a neighbourhood of infinity} \\ &\text{such that } t^{1/\alpha}/\psi(t) \text{ or } t^{1/\alpha}\psi(t) \text{ is nondecreasing} \\ &\text{in a neighborhood of infinity for some } \alpha > 0\}. \end{aligned} \tag{1.1}$$

Define the following integrals,

$$\mathcal{I}_d(\psi, c) = \int^{\infty} \frac{\max\big(1, \psi^d(t)\big)}{t} \exp\big(-c\psi^2(t)\big)\,\mathrm{d}t,\ c > 0,\ d = 0, 1, \ldots, \tag{1.2}$$

and

$$\mathcal{J}_d(\psi) = \int^{\infty} \frac{1}{t\psi^d(t)}\,\mathrm{d}t,\ d = 1, 2, \ldots. \tag{1.3}$$

We will write $\mathcal{I}_d(\psi)$ instead of $\mathcal{I}_d(\psi, 1)$. Define

$$c^*(\psi) = \inf\{c > 0;\ \mathcal{I}_d(\psi, c) < \infty\}, \tag{1.4}$$

and notice that for $\psi \in \Psi$ the definition of $c^*(\psi)$ does not depend on d.

2. Limsup and liminf behaviour of the Wiener process

Let $\{W(t);\ t \geq 0\}$ be a standard one-dimensional Brownian motion. For $t \geq 0$ define

$$M^+(t) = \sup_{0\le s\le t} W(s), \qquad M^-(t) = -\inf_{0\le s\le t} W(s),$$
$$M(t) = \sup_{0\le s\le t} |W(s)| = \max\,[M^+(t), M^-(t)].$$

In Keprta (1997) we find the following improvement of the Kolmogorov's classical integral test (to keep the things simple we state the theorem only for one-dimensional Brownian motion, for r-dimensional case see Keprta (1997)).

Theorem 2.1. *Let $\psi \in \Psi$. Then*

$$\mathrm{P}\left\{M(t) \geq \sqrt{2}\,t^{1/2}\psi(t)\ \textit{i.o. as}\ t \to \infty\right\} = \begin{cases} 0 \\ 1 \end{cases} \textit{as}\ \mathcal{I}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \tag{2.1}$$

Corollary 2.1.

$$\limsup_{t\to\infty} \frac{M(t)}{t^{1/2}\psi(t)} = \big(2c^*(\psi)\big)^{1/2} \quad \textit{a.s.}$$

Remark 2.1. In the above theorem and corollary $M(t)$ can be replaced by $W(t)$, $-W(t)$, $|W(t)|$, $M^+(t)$, or $M^-(t)$.

Example 2.1. The classical example is this. Let $\psi^2(t) = \log_2 t + \frac{3}{2}\log_3 t + \log_4 t + \cdots + \log_{p-1} t + (1+\delta)\log_p t$, where $\log_2 t = \log(3 \vee \log t)$, $\log_k t = \log_{k-1}(3 \vee \log t)$, $k = 3, \ldots$. Then for $p \geq 4$,

$$\psi(t) \in \begin{cases} \mathrm{UUC}\big((2t)^{-1/2}M(t)\big) \\ \mathrm{ULC}\big((2t)^{-1/2}M(t)\big) \end{cases} \quad \text{acc. as} \quad \begin{cases} \delta > 0 \\ \delta \leq 0. \end{cases}$$

The Kolmogorov's integral test assumes that ψ is nondecreasing. The assumption that ψ is nondecreasing implies that ψ is close to $(\log\log t)^{1/2}$, in particular, if $\psi(t_k) < (1-\varepsilon)(\log\log t_k)^{1/2}$ for a sequence $t_k \nearrow \infty$ and some ε, $0 < \varepsilon < 1$, then the probability in (2.1) is one. However, under our monotonicity assumptions we are able to construct a functions ψ such that

$$\liminf_{t\to\infty} \frac{\psi(t)}{(\log\log t)^{1/2}} = 0,$$

but $\mathcal{I}_1(\psi) < \infty$ and $\sqrt{2}\, t^{1/2}\psi(t) \in \mathrm{UUC}(M(t))$.

Let f be a positive, nondecreasing function from $[3,\infty)$ to $(0,\infty)$, such that $f(t) \to \infty$ as $t \to \infty$, and $f(t) < (2\log\log t)^{1/2}$ for all $t \geq 3$. Let $3 \leq T_1 < T_1' < T_1'' < T_2 < T_2' < T_2'' < \cdots$ and ψ be defined in the following way,

$$\psi(t) = \begin{cases} (2\log\log t)^{1/2}, & T_n < t \leq T_n', \\ \psi(T_n')T_n'^{1/2}t^{-1/2}, & T_n' < t \leq T_n'', \\ f(t), & T_n'' < t \leq T_{n+1}, \end{cases}$$

where

$$T_n'' = \inf\{t > T_n';\ f(t) \geq \psi(T_n')T_n'^{1/2}t^{-1/2}\}.$$

Then $\mathcal{I}_1(\psi) < \infty$ if and only if

$$\sum_n \int_{T_n''}^{T_{n+1}} \frac{\psi(t)}{t} \exp\big(-\psi^2(t)\big)\,\mathrm{d}t = \sum_n \int_{T_n''}^{T_{n+1}} \frac{f(t)}{t} \exp\big(-f^2(t)\big)\,\mathrm{d}t < \infty.$$

Hence, f can go to infinity arbitrarily slowly, and convergence or divergence of the integral is decided on the intervals where ψ is equal to f.

Another interesting example is this. Put $\psi(t) = (2\log\log t)^{1/2}$ for $t \notin \mathbb{N}$ and $\psi(t) = 2$ for $t \in \mathbb{N}$. Function ψ is such that $\mathcal{I}_1(\psi) < \infty$, however, since $\psi(t) \nrightarrow \infty$ as $t \to \infty$, $\sqrt{2}\, t^{1/2}\psi(t) \in \mathrm{ULC}\big(M(t)\big)$. In the same spirit we could define ψ in a continuous way, but it is obvious that $t^{1/\alpha}\psi(t)$ and $t^{1/\alpha}/\psi(t)$ are not nondecreasing for any $\alpha > 0$.

In the same way Keprta (1997) generalizes Chung's and Hirsch's integral tests (again, we limit our presentation to one-dimensional case).

Theorem 2.2. *Let $\psi \in \Psi$. Then*

(2.2)
$$\mathrm{P}\left\{M(t) \le \frac{\pi}{\sqrt{8}} \cdot \frac{t^{1/2}}{\psi(t)} \ \textit{i.o. as } t \to \infty\right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \textit{acc. as } \mathcal{I}_2(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Corollary 2.2.
$$\liminf_{t\to\infty} \frac{M(t)}{t^{1/2}}\psi(t) = \left(\frac{\pi^2}{8c^*(\psi)}\right)^{1/2} \qquad \textit{a.s.}$$

Theorem 2.3. *Let $\psi \in \Psi$. Then*

$$\mathrm{P}\left\{M^+(t) \le \frac{t^{1/2}}{\psi(t)} \ \textit{i.o. as } t \to \infty\right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \textit{acc. as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty, \end{cases} \tag{2.3}$$

or, equivalently,
$$\liminf_{t\to\infty} \frac{M^+(t)}{t^{1/2}}\psi(t) = \begin{cases} \infty \\ 0 \end{cases} \quad \textit{a.s. acc. as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Remark 2.2. An important property of the $\mathcal{I}_d$ and $\mathcal{J}_d$ integrals is that if $\beta > 0$ and $\psi \in \Psi$ then there exist $\psi_1, \psi_2 \in \Psi$ such that $\psi_1 \le \psi \le \psi_2$, both $t^{1/\beta}/\psi_i(t)$ and $t^{1/\beta}\psi_i(t)$ are nondecreasing, $i = 1, 2$, and $\mathcal{I}_d(\psi)$, $\mathcal{I}_d(\psi_1)$, $\mathcal{I}_d(\psi_2)$ ($\mathcal{J}_d(\psi)$, $\mathcal{J}_d(\psi_1)$, $\mathcal{J}_d(\psi_2)$, respectively) are all convergent or all divergent. Therefore, without loss of generality, we can prove the above theorems only for ψ with both $t^{1/\beta}\psi(t)$ and $t^{1/\beta}/\psi(t)$ nondecreasing for, say, $\beta = 2$, and the theorems remain true for all $\psi \in \Psi$.

Define the first passage time process
$$S^+(u,\omega) = \inf\{t \ge 0;\ W(t,\omega) = u\} = \inf\{t \ge 0;\ M^+(t,\omega) = u\}.$$

$\{S^+(u);\ u \ge 0\}$ is an inverse of $\{M^+(t);\ t \ge 0\}$. If f is a positive increasing function and g is its inverse then $f \in \mathrm{UUC}(M^+)$ if and only if $g \in \mathrm{LLC}(S^+)$ and $f \in \mathrm{LLC}(M^+)$ if and only if $g \in \mathrm{UUC}(S^+)$. Since $\{S^+(u);\ u \ge 0\}$ is a nondecreasing process with stationary, independent increments, it is easier to work with $\{S^+(u);\ u \ge 0\}$ than to prove Theorems 2.1 and 2.3 directly, see Keprta (1997) for details.

3. Cauchy process

We will present theorems for the Cauchy process analogous to those mentioned above for the Brownian motion.

$\{\mathsf{C}(t);\ t \ge 0\}$ is a (symmetric one-dimensional) Cauchy process defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ if

i) $\mathsf{C} : [0,\infty) \times \Omega \to \mathbb{R}$;
ii) $\mathsf{C}(0,\omega) = 0$ for each ω;
iii) $\mathsf{C}(t,\cdot)$ is $\mathcal{F}$-measurable for each t;

iv) the increments are stationary, independent, $\mathsf{C}(t) \stackrel{\mathcal{D}}{=} t\mathsf{C}(1)$ and the distribution of $\mathsf{C}(1)$ is given by

$$\mathrm{P}\{\mathsf{C}(1) \le x\} = \frac{1}{2} + \frac{1}{\pi}\arctan x.$$

v) $\mathsf{C}(\cdot, \omega)$ is right-continuous and has finite left-hand side limits for almost all ω.

The Cauchy process can be represented by two independent Brownian motions. Take $\{W^{(1)}(t);\ t \ge 0\}$ and $\{W^{(2)}(t);\ t \ge 0\}$ two independent Brownian motions and let $\{S^+(u);\ u \ge 0\}$ be the first passage time process corresponding to $W^{(1)}(t)$. Then $\mathsf{C}(t) = W^{(2)}\big(S^+(t)\big)$, $t \ge 0$, is a standard Cauchy process. The Cauchy process is neighbourhood recurrent but not point recurrent.

Define

$$\mathsf{C}^+(t) = \sup_{0\le s\le t} \mathsf{C}(s), \quad t \ge 0, \qquad \mathsf{C}^-(t) = -\inf_{0\le s\le t} \mathsf{C}(s), \quad t \ge 0,$$

$$\mathsf{C}^*(t) = \sup_{0\le s\le t} |\mathsf{C}(s)| = \max\,[\mathsf{C}^+(t), \mathsf{C}^-(t)], \quad t \ge 0.$$

Since $\{\mathsf{C}(t);\ t \ge 0\} \stackrel{\mathcal{D}}{=} \{-\mathsf{C}(t);\ t \ge 0\}$, also $\{C^+(t);\ t \ge 0\} \stackrel{\mathcal{D}}{=} \{\mathsf{C}^-(t);\ t \ge 0\}$. We will study lim sup and lim inf behaviour of these processes as $t \to \infty$.

4. Limsup behaviour of C

Bertoin (1996) Theorem 5, p. 222, gives a general theorem for stable processes (see also Fristedt (1974)). We slightly enlarge the class of the weight functions and state the theorem for the Cauchy process specifically.

Theorem 4.1. *Let $\psi \in \Psi$. Then*

$$\mathrm{P}\left\{\mathsf{C}(t) \ge t\psi(t) \text{ i.o. as } t \to \infty\right\} = \begin{cases} 0 \\ 1 \end{cases} \text{acc. as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases} \tag{4.1}$$

Corollary 4.1.

$$\limsup_{t\to\infty} \frac{\mathsf{C}(t)}{t\psi(t)} = \begin{cases} 0 \\ \infty \end{cases} \text{a.s. acc. as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Remark 4.1. In the above theorem and corollary $\mathsf{C}(t)$ can be replaced by $-\mathsf{C}(t)$, $|\mathsf{C}(t)|$, $\mathsf{C}^+(t)$, $\mathsf{C}^-(t)$, or $\mathsf{C}^*(t)$.

The following generalization of the Borel–Cantelli lemma will be used in our proofs.

Lemma 4.1. *Let B_k, $k = 1, \dots,$ be arbitrary events. If $\sum_k \mathrm{P}(B_k) < \infty$, then $\mathrm{P}(B_k$ i.o.$) = 0$. If $\sum_k \mathrm{P}(B_k) = \infty$ and*

$$\liminf_{n\to\infty} \frac{\sum\limits_{k=1}^{n}\sum\limits_{l=1}^{n} \mathrm{P}(B_k \cap B_l)}{\Big(\sum\limits_{k=1}^{n} \mathrm{P}(B_k)\Big)^2} \le C \quad (C \ge 1),$$

then $\mathrm{P}(B_k$ i.o.$) \ge 1/C$.

Proof of Theorem 4.1. Without loss of generality we may assume that $t\psi(t)$ is nondecreasing. Assume that $\mathcal{J}_1(\psi) < \infty$. Then $\psi(t) \to \infty$ as $t \to \infty$. Denote

$$A_k = \{\mathsf{C}^+(t) \ge t\psi(t) \text{ for some } t \in [2^k, 2^{k+1})\}.$$

Since both $\mathsf{C}^+(t)$ and $t\psi(t)$ are nondecreasing, we have

$$\mathrm{P}(A_k) \le \mathrm{P}\{\mathsf{C}^+(2^{k+1}) \ge 2^k\psi(2^k)\} = \mathrm{P}\{\mathsf{C}^+(1) \ge \tfrac{1}{2}\psi(2^k)\} \approx \frac{1}{\psi(2^k)},$$

where $a(x) \approx b(x)$ as $x \to L$ means that $c \le \liminf_{x\to L} a(x)/b(x) \le \limsup_{x\to L} a(x)/b(x) \le c'$ for some positive constants c and c'. The last approximation follows from $\mathrm{P}\{\mathsf{C}^+(1) \ge x\} \approx \mathrm{P}\{\mathsf{C}(1) \ge x\} \approx 1/x$ for $x \to \infty$. It is easy to show that for $\lambda > 1$, $\mathcal{J}_1(\psi) < \infty$ if and only if $\sum_k 1/\psi(\lambda^k) < \infty$. Therefore $\sum_k \mathrm{P}(A_k) < \infty$ and the Borel–Cantelli lemma finishes the proof of the convergent part.

Now assume that $\mathcal{J}_1(\psi) = \infty$. The case $\psi(t) \nrightarrow \infty$ as $t \to \infty$ is obvious, hence assume $\psi(t) \to \infty$ as $t \to \infty$. Define for $a > 0$

$$A_k = \{\mathsf{C}(2^{k+1}) - \mathsf{C}(2^k) \ge 2^{k+1}\psi(2^{k+1}) + a2^k, |\mathsf{C}(2^k)| \le a2^k\}.$$

Obviously A_k implies $\mathsf{C}(2^{k+1}) \ge 2^{k+1}\psi(2^{k+1})$. Using the independence and stationarity of the increments and the scaling property, we have

$$\mathrm{P}(A_k) = \mathrm{P}\{|\mathsf{C}(1)| \le a\}\,\mathrm{P}\{\mathsf{C}(1) > 2\psi(2^{k+1}) + a\}.$$

Therefore, $\mathrm{P}(A_k) \approx 1/\psi(2^{k+1})$ and $\sum_k \mathrm{P}(A_k) = \infty$. Moreover, for $k \ne l$,

$$\mathrm{P}(A_k \cap A_l) \le \mathrm{P}(A_k)\,\mathrm{P}(A_l)/\big(\mathrm{P}\{|\mathsf{C}(1)| \le a\}\big)^2.$$

The Borel–Cantelli lemma and the zero-one law gives $\mathrm{P}(A_k \text{ i.o.}) = 1$. This finishes the proof of the theorem. □

5. Liminf behaviour of $|\mathsf{C}|$

Now we will slightly generalize Theorem 11.5 of Fristedt (1974) and state it as a special case for the Cauchy process. This is similar to Spitzer's theorem for planar Brownian motion (see Spitzer (1958)).

Theorem 5.1. *Let $\psi \circ \exp \in \Psi$. Then*

$$\liminf_{t\to\infty} |\mathsf{C}(t)| \exp\bigl(\psi(t)\log t\bigr) = \begin{cases} \infty \\ 0 \end{cases} \quad \text{a.s. acc. as } \mathcal{J}_0(\psi) \begin{cases} < \infty \\ = \infty, \end{cases} \tag{5.1}$$

where

$$\mathcal{J}_0(\psi) = \int^{\infty} \frac{1}{t\psi(t)\log t}\,\mathrm{d}t.$$

Proof. Take $\lambda > 1$ and put

$$A_k = \{|\mathsf{C}(t)| \le h(t) \text{ for some } t \in [\lambda^k, \lambda^{k+1})\},$$

where $h(t) = \exp\bigl(-\psi(t)\log t\bigr)$. Then $\sum_k \mathrm{P}(A_k)$ converges or diverges according as $\mathcal{J}_0(\psi)$ converges or diverges. For $l > k+1$ it is possible to show that

$$\mathrm{P}(A_k \cap A_l) \le C\,\mathrm{P}(A_k)\,\mathrm{P}(A_l),$$

for some positive constant C. An application of the Borel–Cantelli lemma finishes the proof. □

6. Liminf behaviour of C^+

We will prove the following theorem.

Theorem 6.1. *Let $\psi \in \Psi$. Then*

$$\mathrm{P}\left\{\mathsf{C}^+(t) \le \frac{t}{\psi^2(t)} \text{ i.o. as } t \to \infty\right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{acc. as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty, \end{cases} \tag{6.1}$$

or, equivalently,

$$\liminf_{t\to\infty} \frac{\mathsf{C}^+(t)}{t}\psi^2(t) = \begin{cases} \infty \\ 0 \end{cases} \quad \text{a.s. acc. as } \mathcal{J}_1(\psi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Put

$$\mathsf{T}^+(u) = \inf\{t \ge 0;\ \mathsf{C}^+(t) \ge u\}.$$

Then

$$\{\mathsf{C}^+(t) \le u\} = \{\mathsf{T}^+(u) \ge t\}$$

and $\{\mathsf{T}^+(u);\ u \ge 0\}$ has similar properties as $\{S^+(u);\ u \ge 0\}$, namely it has nondecreasing paths and has stationary, independent increments, hence, it is a subordinator (see Karatzas and Shreve (1988) for basic properties of subordinators and Bertoin (1996) for integral tests for subordinators). Since $\mathsf{C}^+(t)/t \stackrel{\mathcal{D}}{=} \mathsf{C}^+(1)$ for each $t > 0$, we have $\mathsf{T}^+(u)/u \stackrel{\mathcal{D}}{=} \mathsf{T}^+(1)$ for each $u > 0$. Since

$$\mathrm{P}\{\mathsf{C}^+(1) \le x\} \approx \sqrt{x}, \quad x \to 0+,$$

(see Darling (1956)), we have

$$\mathrm{P}\{\mathsf{T}^+(1) \geq \frac{1}{x}\} \approx \sqrt{x}, \quad x \to 0+.$$

Put

$$u = u(t) = t/\psi^2(t)$$

and assume that $u(t)$ is strictly increasing. Then its inverse has a form

$$t = t(u) = u\xi^2(u),$$

where $\xi(u) = \psi(t)$. It is not difficult to show that

$$\mathcal{J}_1(\psi) < \infty \iff \mathcal{J}_1(\xi) < \infty.$$

Since in Theorem 6.1 it is possible to assume without loss of generality that $t/\psi^2(t)$ is strictly increasing, Theorem 6.1 is equivalent to

Theorem 6.2. *Let* $\xi \in \Psi$. *Then*

$$\text{(6.2)} \quad \mathrm{P}\left\{\mathsf{T}^+(u) \geq u\xi^2(u) \textit{ i.o. as } u \to \infty\right\} = \begin{cases} 0 \\ 1 \end{cases} \textit{acc. as } \mathcal{J}_1(\xi) \begin{cases} < \infty \\ = \infty, \end{cases}$$

or, equivalently,

$$\limsup_{u\to\infty} \frac{\mathsf{T}^+(u)}{u\xi^2(u)} = \begin{cases} 0 \\ \infty \end{cases} \textit{a.s. acc. as } \mathcal{J}_1(\xi) \begin{cases} < \infty \\ = \infty. \end{cases}$$

Proof. We may assume without loss of generality that $u\xi^2(u)$ is strictly increasing. Assume that $\mathcal{J}_1(\xi) < \infty$. This implies that $\xi(u) \to \infty$ as $u \to \infty$. Denote

$$A_k = \{\mathsf{T}^+(u) \geq u\xi^2(u) \text{ for some } u \in [2^k, 2^{k+1})\}.$$

Since both $\mathsf{T}^+(u)$ and $u\xi^2(u)$ are nondecreasing, we have

$$\mathrm{P}(A_k) \leq \mathrm{P}\{\mathsf{T}^+(2^{k+1}) \geq 2^k\xi^2(2^k)\} = \mathrm{P}\{\mathsf{T}^+(1) \geq \tfrac{1}{2}\xi^2(2^k)\} \approx \frac{1}{\xi(2^k)}.$$

It is easy to show that for $\lambda > 1$, $\mathcal{J}_1(\xi) < \infty \iff \sum_k 1/\xi(\lambda^k) < \infty$. Therefore $\sum_k \mathrm{P}(A_k) < \infty$ and by the Borel–Cantelli lemma $\mathrm{P}(A_k \text{ i.o.}) = 0$.

Now assume that $\mathcal{J}_1(\xi) = \infty$. The case $\xi(u) \not\to \infty$ as $u \to \infty$ is obvious, hence assume $\xi(u) \to \infty$ as $u \to \infty$. Define

$$A_k = \{\mathsf{T}^+(2^k) \geq 2^k\xi^2(2^k)\}$$

and

$$B_k = \{\mathsf{T}^+(2^k) - \mathsf{T}^+(2^{k-1}) \geq 2^k\xi^2(2^k)\}.$$

Obviously $B_k \subset A_k$. To show that $\mathrm{P}(A_k \text{ i.o.}) = 1$, it is sufficient to show that $\mathrm{P}(B_k \text{ i.o.}) = 1$. Since B_k, $k = 1, 2 \ldots$, are independent, it is enough to show that $\sum_k \mathrm{P}(B_k) = \infty$. Since $\mathsf{T}^+(2^k) - \mathsf{T}^+(2^{k-1}) \stackrel{\mathcal{D}}{=} \mathsf{T}^+(2^{k-1})$, we have

$$\mathrm{P}(B_k) = \mathrm{P}\{\mathsf{T}^+(2^{k-1}) \geq 2^k \xi^2(2^k)\} = \mathrm{P}\{\mathsf{T}^+(1) \geq 2\xi^2(2^k)\} \approx \frac{1}{\xi(2^k)},$$

the sum of which diverges. This finishes the proof of the theorem. □

7. Liminf behaviour of C^*

The following theorem characterizes the lower classes of $\{\mathsf{C}^*(t);\ t \geq 0\}$.

Theorem 7.1. *Let $\psi \in \Psi$. Then*

(7.1)
$$\mathrm{P}\left\{\mathsf{C}^*(t) \leq \frac{\pi}{2\lambda_1} \cdot \frac{t}{\psi^2(t)} \ \textit{i.o. as } t \to \infty\right\} = \begin{cases} 0 \\ 1 \end{cases} \quad \textit{acc. as } \mathcal{I}_2(\psi) \begin{cases} < \infty \\ = \infty, \end{cases}$$

where $\lambda_1 \doteq 1.36$ denotes the largest eigenvalue of the kernel

$$K(x,y) = \sum_{m=1}^{\infty} \frac{\sin(m \arccos x)\sin(m \arccos y)}{m}.$$

Remark 7.1. A function f and a number λ is an eigenfunction and an eigenvalue of a kernel K if $Kf = \lambda f$, where the operator K is defined as $Kf = \int_{-1}^{1} K(x,y) f(x)\,\mathrm{d}x$.

Corollary 7.1.

$$\liminf_{t\to\infty} \frac{\mathsf{C}^*(t)}{t} \psi^2(t) = \frac{\pi}{2\lambda_1 c^*(\psi)} \quad \textit{a.s.}$$

Proof. We may assume without loss of generality that $t\psi^2(t)$ and $t/\psi^2(t)$ are strictly increasing, and that $2 \leq \psi^2(t) \leq 2\log\log t$ for all t. Let t_0 be an initial value and define

$$t_{k+1} = t_k \left(1 + \frac{1}{\psi_k^2 - 1}\right) = t_k \frac{\psi_k^2}{\psi_k^2 - 1},$$

where $\psi_k = \psi(t_k)$. Then $\mathcal{I}_2(\psi) < \infty$ if and only if $\sum_k \exp(-\psi_k^2) < \infty$.

Let $\lambda_1 \geq \lambda_2 \geq \cdots$ denote the eigenvalues and g_j, $j = 1, 2, \ldots$, the corresponding eigenfunctions of the kernel $K(x,y)$. Then (see Kac and Pollard (1950))

(7.2)
$$G_r(x) = \mathrm{P}\left\{\frac{\mathsf{C}^*(t)}{t} < x\right\} = \sum_{j=1}^{\infty} \exp\left(-\frac{\pi}{2x\lambda_j}\right) g_j(0) \int_{-1}^{1} g_j(y)\,\mathrm{d}y.$$

Since $\lambda_1 > \lambda_2$, the first term of the sum in (7.2) is dominant and we have

(7.3)
$$G_r(x) \approx \exp\left(-\frac{\pi}{2x\lambda_1}\right) \quad \text{for } x \to 0+.$$

Assume that $\mathcal{I}_2(\psi) < \infty$. Then $\psi(t) \to \infty$ as $t \to \infty$. Denote

$$A_k = \left\{ \mathsf{C}^*(t) \le \frac{\pi}{2\lambda_1} \cdot \frac{t}{\psi^2(t)} \text{ for some } t \in [t_k, t_{k+1}) \right\}.$$

Since both $\mathsf{C}^*(t)$ and $t/\psi^2(t)$ are nondecreasing, we obtain

(7.4)

$$\begin{aligned} \mathrm{P}(A_k) &\le \mathrm{P}\left\{ \mathsf{C}^*(t_k) \le \frac{\pi}{2\lambda_1} \cdot \frac{t_{k+1}}{\psi_{k+1}^2} \right\} = \mathrm{P}\left\{ \frac{\mathsf{C}^*(t_k)}{t_k} \le \frac{\pi}{2\lambda_1} \cdot \frac{1}{\psi_{k+1}^2} \cdot \frac{t_{k+1}}{t_k} \right\} \\ &\approx \exp\left(-\psi_{k+1}^2 \frac{t_k}{t_{k+1}} \right) = \exp\left(-\frac{\psi_{k+1}^2(\psi_k^2 - 1)}{\psi_k^2} \right) \\ &\le \exp\left(-\frac{(\psi_k^2 - 1)^2}{\psi_k^2} \right) \approx \exp\bigl(-\psi_k^2\bigr), \end{aligned}$$

which sums. Therefore, by the Borel–Cantelli lemma, $\mathrm{P}(A_k \text{ i.o.}) = 0$, which proves the convergent part of the theorem.

Now assume that $\mathcal{I}_2(\psi) = \infty$. The case $\psi(t) \nrightarrow \infty$ as $t \to \infty$ is obvious and, therefore, assume that $\psi(t) \to \infty$ as $t \to \infty$. Define $\mathcal{K} = \{k;\ \psi_k^2 > \psi_{k-1}^2 + \frac{1}{2}\}$ and its complement $\mathcal{K}^c = \mathbb{N} - \mathcal{K}$. Now denote

$$A_k = \left\{ \mathsf{C}^*(t_k) \le \frac{\pi}{2\lambda_1} \cdot \frac{t_k}{\psi_k^2} \right\},$$

and notice that

$$\mathrm{P}(A_k) \approx \exp\bigl(-\psi_k^2\bigr). \tag{7.5}$$

We will use the Borel–Cantelli lemma to show that A_k, $k \in \mathcal{K}^c$, occur infinitely often with probability one. $\mathcal{I}_2(\psi) = \infty$ implies that $\sum_{k=1}^{\infty} \mathrm{P}(A_k) = \infty$. Moreover, it is possible to show that $\sum_{k \in \mathcal{K}^c} \mathrm{P}(A_k) = \infty$.

Let $k < l$, $k, l \in \mathcal{K}^c$. Now

$$\mathrm{P}(A_k \cap A_l) \le \mathrm{P}(A_k)\, \mathrm{P}\left\{ \mathsf{C}^*(t_l - t_k) \le \frac{\pi}{2\lambda_1} \cdot \frac{t_l}{\psi_l^2} \right\},$$

and, by (7.3), we have

$$\mathrm{P}(A_k \cap A_l) \le C \exp\bigl(-\psi_k^2\bigr) \exp\left(-\psi_l^2 \frac{t_l - t_k}{t_l} \right), \tag{7.6}$$

for some positive constant C independent of k and l. For a fixed k,

$$\begin{aligned} \frac{\mathrm{P}(A_k \cap A_l)}{\mathrm{P}(A_k)\,\mathrm{P}(A_l)} &\le C_1 \frac{\exp\bigl(-\psi_k^2\bigr) \exp\left(-\psi_l^2 \frac{t_l - t_k}{t_l} \right)}{\exp\bigl(-\psi_k^2\bigr) \exp\bigl(-\psi_l^2\bigr)} \\ &= C_1 \exp\left(\psi_l^2 \frac{t_k}{t_l} \right) \searrow C_1, \end{aligned}$$

for some positive constant C_1 as $l \to \infty$, since $\psi_l^2/t_l \searrow 0$ as $l \to \infty$.

Define $l_0 = l_0(k)$ by

$$l_0 = \sup \left\{ l;\ l > k,\ \psi_l^2 \frac{t_k}{t_l} > 1 \right\}.$$

Then for $l > l_0$,

$$\frac{\mathrm{P}(A_k \cap A_l)}{\mathrm{P}(A_k)\,\mathrm{P}(A_l)} \le C_1 e.$$

We want to show that there is a constant C_2, independent of k, such that

$$\sum_{\substack{l=k+1 \\ l \in \mathcal{K}^c}}^{l_0} \frac{\mathrm{P}(A_k \cap A_l)}{\mathrm{P}(A_k)} \le C_2. \tag{7.7}$$

It is possible to show that for $l > k$, $l \in \mathcal{K}^c$,

$$\begin{aligned} \frac{\mathrm{P}(A_k \cap A_l)}{\mathrm{P}(A_k)} &\le C_3 \exp\left(-\psi_l^2 \frac{t_l - t_k}{t_l} \right) \\ &\le K \int_{z_l}^{z_{l-1}} \frac{1}{z} e^z \xi^2(z) \exp\bigl(-\xi^2(z)\bigr)\,\mathrm{d}z, \end{aligned} \tag{7.8}$$

where K is a universal constant and $z = z(t) = t_k \psi^2(t)/t$, $z_l = t_k \psi_l^2/t_l$, and $\xi(z) = \psi(t)$. Notice that $t\psi^2(t) = t_k \xi^4(z)/z$ is nondecreasing as t increases, ie., as z decreases.

Observe that $\xi^2(z_k) = \psi_k^2$ and $z_{l_0} > 1$. From (7.8) we have

$$\begin{aligned} \sum_{\substack{l=k+1 \\ l \in \mathcal{K}^c}}^{l_0} \frac{\mathrm{P}(A_k \cap A_l)}{\mathrm{P}(A_k)} &\le K \sum_{\substack{l=k+1 \\ l \in \mathcal{K}^c}}^{l_0} \int_{z_l}^{z_{l-1}} \frac{1}{z} e^z \xi^2(z) \exp\bigl(-\xi^2(z)\bigr)\,\mathrm{d}z \\ &\le K \int_1^{\psi_k^2} \frac{1}{z} e^z \xi^2(z) \exp\bigl(-\xi^2(z)\bigr)\,\mathrm{d}z \\ &\le K \int_1^{\psi_k^2} \frac{1}{z} e^z \psi_k \sqrt{z} \exp\bigl(-\psi_k \sqrt{z}\bigr)\,\mathrm{d}z \\ &\le 3K = C_2, \end{aligned}$$

where we replaced $\xi^2(z)$ by $\psi_k \sqrt{z}$, the smallest possible function satisfying $\xi^2(\psi_k^2) = \psi_k^2$ and $\xi^2(z)/\sqrt{z}$ is nonincreasing, and, since xe^{-x} is decreasing for $x > 1$, made the integral as big as possible.

Now it is easy to verify that

$$\liminf_{N\to\infty} \frac{\displaystyle\sum_{\substack{k=1\\k\in\mathcal{K}^c}}^{N} \sum_{\substack{l=1\\l\in\mathcal{K}^c}}^{N} \mathrm{P}(A_k \cap A_l)}{\left[\displaystyle\sum_{\substack{k=1\\k\in\mathcal{K}^c}}^{N} \mathrm{P}(A_k)\right]^2} \le C_4 < \infty$$

for some positive constant C_4, which, by the Borel–Cantelli lemma and zero-one law finishes the proof of the divergent part of the theorem. □

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